

From a dissertation submitted in partial fulfillment of the requirements for the degree of Doctor
of Science in the University of Michigan

ANALYSIS OF STRESS IN CIRCULAR ELASTIC RINGS UNDER THE ACTION OF BODY FORCES AND EXTERNAL LOADING

by

WITOLD BILLEWICZ, C.E., Sc. D.

Presented at the Academy of Technical Sciences Meeting
Warsaw, December 1-st, 1933



WARSZAWA — 1934

WYDANE Z ZAPOMOGI AKADEMJI NAUK TECHNICZNYCH



Digitized by the Internet Archive
in 2026

https://archive.org/details/IA41549008_0185

From a dissertation submitted in partial fulfillment of the requirements for the degree of Doctor
of Science in the University of Michigan

ANALYSIS OF STRESS IN CIRCULAR ELASTIC RINGS UNDER THE ACTION OF BODY FORCES AND EXTERNAL LOADING

by

WITOLD BILLEWICZ, C.E., Sc. D.

Presented at the Academy of Technical Sciences Meeting
Warsaw, December 1-st, 1933



WARSZAWA — 1934

WYDANE Z ZAPOMOZI AKADEMJI NAUK TECHNICZNYCH

DRUKARNIA SPOŁECZNA
WARSAWA
PL. GRZYBOWSKI 3/5 TEL. 205-80



CONTENTS.

	Page
Short Bibliography	5
I. General equations of the theory of elasticity for a two dimensional problem .	7
II. General solution of stress and strain problems in circular elastic rings under the action of body forces and external loading	10
III. Stresses in a circular ring under the action of body forces and a single concen- trated force P	17
IV. Stresses in a circular ring, compressed by two opposite forces acting along a diameter	31
V. Appendix	
Stress distribution in circular arches under the action of pure bending .	43
Shearing stress distribution in circular arches under the action of a force P in a radial cross section	47
Conclusion	51

SHORT BIBLIOGRAPHY.

- 1) S. Timoshenko. "Theory of Elasticity" (Manuscript) 1931.
 - 2) S. Timoshenko. Biulletin Kiev Polytechnic Institute. "Compression of a ring by two equal opposite forces".
 - 3) M. T. Huber. "Probleme der Statik technisch wichtiger orthotroper Platten". Warszawa, 1929.
 - 4) A. E. H. Love. "A Treatise on the Mathematical Theory of Elasticity". 1906.
 - 5) A. u. L. Föppl. "Drang und Zwang".
 - 6) A. Timpe. "Zeitschrift für Mathematik und Physik". 1905.
 - 7) J. H. Michell. "Proceedings on the London Mathematical Society". 1900.
 - 8) E. Kohl. "Ingenieur Archiv". 1930.
 - 9) W. E. Byerly. "Fourier's Series and Spherical Harmonics". 1893.
 - 10) C. Runge. "Theorie und Praxis der Reihen". 1904.
-

I. General equations of the theory of elasticity for a two dimensional problem.

The present work considers the problem of stress distribution in circular rings under the action of body forces and external loading, as given in the form of concentrated forces or as any continuous loading over the external and internal boundaries of the ring. By body forces is meant here the weight of the ring, acting vertically.

The exact solution of the problem will be given by using the mathematical theory of elasticity and a subsequent comparison of the results obtained with those of the elementary solution usually applicable for practical purposes.

The circular ring may be considered as a two dimensional structure of 2 different types:

- 1) with two dimensional strain—when the ring is a part of a tube infinitely long,
- 2) with two dimensional stress—when it consists of a plate infinitely thin.

Conditions of equilibrium of an element for rectangular coordinates as shown in Fig. 1 may be expressed as follows:

$$\left. \begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \gamma_w &= 0 \end{aligned} \right\} \dots (1)$$

where γ_w is the unit weight of material, along the y axis.

The condition of compatibility is given by equation:

$$\frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \dots (2)$$

where e_x , e_y , γ_{xy} represent unit deformations due to Hooke's law.

$$\left. \begin{aligned} e_x &= \frac{\sigma_x}{E} - \mu \frac{\sigma_y}{E} \\ e_y &= \frac{\sigma_y}{E} - \mu \frac{\sigma_x}{E} \\ \gamma_{xy} &= \frac{2 \tau_{xy} (1 + \mu)}{E} \end{aligned} \right\} \dots (3)$$

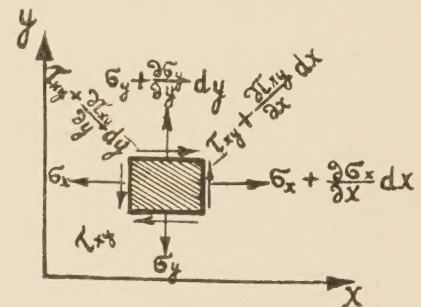


Fig. 1.

E —is the modulus of elasticity, μ — Poisson's ratio.

Values of stresses $\sigma_x, \sigma_y, \tau_{xy}$ may be found from the "stress function" " Φ " as follows

$$\left. \begin{aligned} \sigma_x &= \frac{\partial^2 \Phi}{\partial y^2} \\ \sigma_y &= \frac{\partial^2 \Phi}{\partial x^2} \\ \tau_{xy} &= -\frac{\partial^2 \Phi}{\partial x \partial y} - \gamma_w \cdot x \end{aligned} \right\} \dots \dots \dots (4)$$

It is easy to see that formulas (4) are in agreement with conditions of equilibrium (1); and equation (2), after the substitution of values, may be written in the form

$$\begin{aligned} \frac{\partial^4 \Phi}{\partial x^4} + 2 \frac{\partial^4 \Phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \Phi}{\partial y^4} &= 0 \dots \dots \dots (5) \\ \text{or } \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} \right) &= 0 \dots \dots (5a) \end{aligned}$$

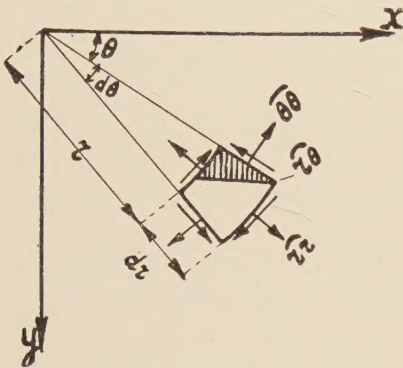


Fig. 2.

To find the stresses shown in (4), equation (5) must first be solved for Φ ; then this value must be substituted into (4) and the conditions of external loading be satisfied.

All previous formulas may be transformed now into the system of polar coordinates.

$$\begin{aligned} \frac{\partial^2 \Phi}{\partial x^2} &= \frac{\partial^2 \Phi}{\partial r^2} \cos^2 \theta - 2 \frac{\partial^2 \Phi}{\partial r \partial \theta} \cdot \frac{\sin \theta \cdot \cos \theta}{r} + 2 \frac{\partial \Phi}{\partial \theta} \cdot \frac{\sin \theta \cdot \cos \theta}{r^2} + \\ &+ \frac{1}{r} \cdot \frac{\partial \Phi}{\partial r} \sin^2 \theta + \frac{\partial^2 \Phi}{\partial \theta^2} \cdot \frac{\sin^2 \theta}{r^2}; \\ \frac{\partial^2 \Phi}{\partial y^2} &= \frac{\partial^2 \Phi}{\partial r^2} \sin^2 \theta + 2 \frac{\partial^2 \Phi}{\partial r \partial \theta} \cdot \frac{\sin \theta \cdot \cos \theta}{r} - 2 \frac{\partial \Phi}{\partial \theta} \cdot \frac{\sin \theta \cdot \cos \theta}{r^2} + \\ &+ \frac{1}{r} \cdot \frac{\partial \Phi}{\partial r} \cos^2 \theta + \frac{\partial^2 \Phi}{\partial \theta^2} \cdot \frac{\cos^2 \theta}{r^2}; \\ \frac{\partial^2 \Phi}{\partial x \partial y} &= \frac{\partial^2 \Phi}{\partial r^2} \sin \theta \cdot \cos \theta + \frac{\partial^2 \Phi}{\partial r \partial \theta} \cdot \frac{\cos 2 \theta}{r} - \frac{\partial^2 \Phi}{\partial \theta^2} \cdot \frac{\sin \theta \cdot \cos \theta}{r^2} + \\ &- \frac{\partial \Phi}{\partial \theta} \cdot \frac{\cos 2 \theta}{r^2} - \frac{\partial \Phi}{\partial r} \cdot \frac{\sin \theta \cdot \cos \theta}{r} \end{aligned} \quad (6)$$

After the substitution of these values into equation (5a) the condition of compatibility will be obtained in polar coordinates.

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2}{\partial \theta^2} \right) \left(\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 \Phi}{\partial \theta^2} \right) = 0 \quad (6a)$$

To find the relations between $\sigma_x, \sigma_y, \tau_{xy}$ and $\widehat{rr}, \widehat{\theta\theta}, \widehat{r\theta}$, the conditions of equilibrium of an element marked on Fig. 2 will be considered. Values of $\sigma_x, \sigma_y, \tau_{xy}$ will be expressed as functions of $\widehat{rr}, \widehat{\theta\theta}, \widehat{r\theta}$, as follows:

$$\left. \begin{aligned} \sigma_x &= \widehat{rr} \cdot \cos^2 \theta + \widehat{\theta\theta} \cdot \sin^2 \theta - \widehat{r\theta} \cdot \sin 2\theta \\ \sigma_y &= \widehat{\theta\theta} \cdot \cos^2 \theta + \widehat{rr} \cdot \sin^2 \theta + \widehat{r\theta} \cdot \sin 2\theta \\ \tau_{xy} &= \widehat{rr} \cdot \frac{\sin 2\theta}{2} - \widehat{\theta\theta} \cdot \frac{\sin 2\theta}{2} + \widehat{r\theta} \cdot \cos 2\theta \end{aligned} \right\} \dots \dots \dots (7)$$

It may be noticed that

$$\sigma_x + \sigma_y = \widehat{rr} + \widehat{\theta\theta}$$

The inverse transformation of equations (7) gives:

$$\left. \begin{aligned} \widehat{rr} &= \sigma_x \cdot \cos 2\theta + (\sigma_x + \sigma_y) \cdot \sin^2 \theta + \tau_{xy} \cdot \sin 2\theta \\ \widehat{\theta\theta} &= (\sigma_x + \sigma_y) \cdot \cos^2 \theta - \sigma_x \cdot \cos 2\theta - \tau_{xy} \cdot \sin 2\theta \\ \widehat{r\theta} &= (\sigma_x + \sigma_y) \cdot \frac{\sin 2\theta}{2} - \sigma_x \cdot \sin 2\theta + \tau_{xy} \cdot \cos 2\theta \end{aligned} \right\} \dots \dots \dots (8)$$

Or after the substitution of values of σ_x , σ_y , τ_{xy} , from (4) and (6) the final form will be obtained:

$$\left. \begin{aligned} \widehat{rr} &= \frac{1}{r^2} \cdot \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{1}{r} \cdot \frac{\partial \Phi}{\partial r} - \gamma_w \cdot r \cdot \cos \theta \cdot \sin 2\theta \\ \widehat{\theta\theta} &= \frac{\partial^2 \Phi}{\partial r^2} + \gamma_w \cdot r \cdot \cos \theta \cdot \sin 2\theta \\ \widehat{r\theta} &= \frac{1}{r^2} \cdot \frac{\partial \Phi}{\partial \theta} - \frac{1}{r} \cdot \frac{\partial^2 \Phi}{\partial r \cdot \partial \theta} - \gamma_w \cdot r \cdot \cos \theta \cdot \cos 2\theta. \end{aligned} \right\} \dots \dots \dots (9)$$

Values of $\widehat{rr}$, $\widehat{\theta\theta}$, $\widehat{r\theta}$, expressed in this form, will satisfy the conditions of equilibrium of an element $r \cdot dr \cdot d\theta$ as follows:

$$\begin{aligned} \frac{\partial \widehat{rr}}{\partial r} + \frac{1}{r} \cdot \frac{\partial \widehat{r\theta}}{\partial \theta} + \frac{\widehat{rr} - \widehat{\theta\theta}}{r} + \gamma_w \cdot \sin \theta &= 0. \\ \frac{1}{r} \cdot \frac{\partial \widehat{\theta\theta}}{\partial \theta} + \frac{\partial \widehat{r\theta}}{\partial r} + \frac{2 \widehat{r\theta}}{r} + \gamma_w \cdot \cos \theta &= 0. \end{aligned}$$

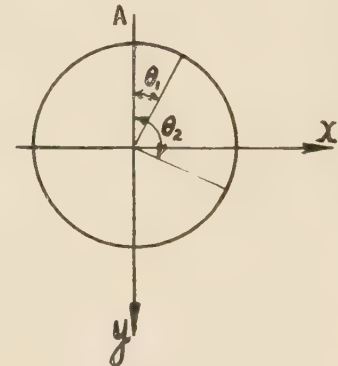


Fig. 3.

For the case of symmetrical loading, where the y axis is the axis of symmetry (Fig. 3), it is more convenient to measure angle θ from the point A , then in formulas (9) the angle $\theta - \frac{\pi}{2}$ should be substituted instead of θ and formulas (9) will take the following form:

$$\left. \begin{aligned} \widehat{rr} &= \frac{1}{r^2} \cdot \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{1}{r} \cdot \frac{\partial \Phi}{\partial r} + \gamma_w \cdot r \cdot \sin \theta \cdot \sin 2\theta \\ \widehat{\theta\theta} &= \frac{\partial^2 \Phi}{\partial r^2} - \gamma_w \cdot r \cdot \sin \theta \cdot \sin 2\theta \\ \widehat{r\theta} &= \frac{1}{r^2} \cdot \frac{\partial \Phi}{\partial \theta} - \frac{1}{r} \cdot \frac{\partial^2 \Phi}{\partial r \cdot \partial \theta} + \gamma_w \cdot r \cdot \sin \theta \cdot \cos 2\theta. \end{aligned} \right\} \dots \dots \dots (10)$$

It should be noticed that terms with γ_w express the effect of body forces on the distribution of internal stresses. These terms may be transformed by using simple trigonometrical relations:

$$\left. \begin{aligned} \sin \Theta \cdot \sin 2 \Theta &= \frac{1}{2} (\cos \Theta - \cos 3 \Theta) \\ \sin \Theta \cdot \cos 2 \Theta &= -\frac{1}{2} (\sin \Theta - \sin 3 \Theta) \end{aligned} \right\} \dots \dots \dots (11)$$

This means that the effect of body forces on the distribution of internal stresses is given by terms with sines and cosines of the angles Θ and 3Θ only. As follows from further considerations, the functions of 3Θ may be omitted, and these terms will be called "silent terms". Thus only terms with sines and cosines of single angle Θ express the effect of body forces on the values of internal stresses.

II. General solution of stress and strain problems in circular elastic rings under the action of body forces and external loading.

The general solution of Airy's function (6a) may be written in the following form¹⁾:

$$\begin{aligned} \Phi &= a_0 \log r + b_0 r^2 + c_0 r^2 \log r + \alpha_0 \Theta + \frac{a_1}{2} r \Theta \sin \Theta + (b_1 r^3 + \alpha_1 r^{-1} + \beta_1 r \log r) \cos \Theta + \\ &\quad - \frac{c_1}{2} r \Theta \cos \Theta + (d_1 r^3 + \gamma_1 r^{-1} + \delta_1 r \log r) \sin \Theta + \\ &\quad + \sum_{n=2}^{\infty} (a_n r^n + b_n r^{n+2} + \alpha_n r^{-n} + \beta_n r^{-n+2}) \cos n \Theta + \\ &\quad + \sum_{n=2}^{\infty} (c_n r^n + d_n r^{n+2} + \gamma_n r^{-n} + \delta_n r^{-n+2}) \sin n \Theta. \quad \dots \dots \dots (12) \end{aligned}$$

From the "stress function" Φ , values of stresses $\widehat{rr}$, $\widehat{\Theta\Theta}$, $\widehat{r\Theta}$, may be found by using the formulas (10).

$$\left. \begin{aligned} \widehat{rr} &= \frac{a_0}{r^2} + 2 b_0 + c_0 (2 \log r + 1) + \left(\frac{\alpha_1 + \beta_1}{r} + 2 b_1 r - \frac{2 \alpha_1}{r^3} \right) \cos \Theta + \\ &\quad + \left(\frac{c_1 + \delta_1}{r} + 2 d_1 r - \frac{2 \gamma_1}{r^3} \right) \sin \Theta + \frac{1}{2} \gamma_w r (\cos \Theta - \cos 3 \Theta) + \\ &\quad + \sum_{n=2}^{\infty} [n(1-n)a_n r^{n-2} + (n+2-n^2)b_n r^n - n(1+n)\alpha_n r^{-n-2} - (n-2+n^2)\beta_n r^{-n}] \cos n \Theta + \\ &\quad + \sum_{n=2}^{\infty} [n(1-n)c_n r^{n-2} + (n+2-n^2)d_n r^n - n(1+n)\gamma_n r^{-n-2} - \delta_n (n-2+n^2) r^{-n}] \sin n \Theta; \end{aligned} \right\} (13a)$$

¹⁾ J. H. Michell, "Proceedings of the London Mathematical Society", 1900, page 100.
A. Timpe, "Zeitschrift für Mathematik und Physik", 1905, page 370.

$$\left. \begin{aligned} \widehat{\Theta\Theta} = & -\frac{a_0}{r^2} + 2b_0 + c_0(2\log r + 3) + \left(6b_1r + \frac{2\alpha_1}{r^3} + \frac{\beta_1}{r}\right) \cos \Theta + \\ & + \left(6d_1r + \frac{2\gamma_1}{r^3} + \frac{\delta_1}{r}\right) \sin \Theta - \frac{1}{2}\gamma_w r (\cos \Theta - \cos 3\Theta) + \\ & + \sum_{n=2}^{\infty} [(n-1)n a_n r^{n-2} + (n+1)(n+2)b_n r^n + n(n+1)\alpha_n r^{-n-2} + (n-2)(n-1)\beta_n r^{-n}] \cos n\Theta + \\ & + \sum_{n=2}^{\infty} [(n-1)n c_n r^{n-2} + (n+1)(n+2)d_n r^n + n(n+1)\gamma_n r^{-n-2} + (n-2)(n-1)\delta_n r^{-n}] \sin n\Theta; \end{aligned} \right\} (13b)$$

$$\left. \begin{aligned} \widehat{r\Theta} = & \frac{\alpha_0}{r^2} + \left(2b_1r - \frac{2\alpha_1}{r^3} + \frac{\beta_1}{r}\right) \sin \Theta - \left(2d_1r - \frac{2\gamma_1}{r^3} + \frac{\delta_1}{r}\right) \cos \Theta + \\ & - \frac{1}{2}\gamma_w r (\sin \Theta - \sin 3\Theta) + \\ & + \sum_{n=2}^{\infty} [(n-1)n a_n r^{n-2} + n(n+1)b_n r^n - n(n+1)\alpha_n r^{-n-2} - n(n-1)\beta_n r^{-n}] \sin n\Theta + \\ & - \sum_{n=2}^{\infty} [(n-1)n c_n r^{n-2} + n(n+1)d_n r^n - n(n+1)\gamma_n r^{-n-2} - (n-1)n\delta_n r^{-n}] \cos n\Theta. \end{aligned} \right\} (13c)$$

To find the values of unknown factors $a_n, b_n, c_n, d_n, \alpha_n, \beta_n, \gamma_n, \delta_n$, the conditions of external loading will be considered (details for $n = 1$ are given in the discussion further on). Any external loading may be determined in the form of components in a radial and a tangential direction as acting on the outer or inner boundary of the ring. After development of these functions of the angle Θ into Fourier's series, the following formulas will be obtained:

$$\left. \begin{aligned} (\widehat{rr})_{r=R_1} &= A_0 + \sum_{n=1}^{\infty} A_n \cos n\Theta + \sum_{n=1}^{\infty} B_n \sin n\Theta. \\ (\widehat{r\Theta})_{r=R_1} &= C_0 + \sum_{n=1}^{\infty} C_n \cos n\Theta + \sum_{n=1}^{\infty} D_n \sin n\Theta. \\ (\widehat{rr})_{r=R_2} &= A'_0 + \sum_{n=1}^{\infty} A'_n \cos n\Theta + \sum_{n=1}^{\infty} B'_n \sin n\Theta. \\ (\widehat{r\Theta})_{r=R_2} &= C'_0 + \sum_{n=1}^{\infty} C'_n \cos n\Theta + \sum_{n=1}^{\infty} D'_n \sin n\Theta. \end{aligned} \right\} \dots \dots (14)$$

where R_1 and R_2 are the radii of the inner and the outer boundaries of the ring.

It is easy to notice that to each value of $n > 1$ substituted into formulas (14) and (13a) or (13c) with $r = R_1$ or $r = R_2$, for the inner and outer boundary of the ring respectively, the coefficients of similar trigonometrical functions in these formulas correspond to each other. This way eight additional equations will be obtained to determine the values of factors $a_n, b_n, c_n, d_n, \alpha_n, \beta_n, \gamma_n, \delta_n$, where $n > 1$.

Values of factors for $n=0$ and $n=1$.

The continuity of distortions in the ring must here be taken into account.

In circular rings without any artificial distortions¹⁾ (which are possible in multiply-connected regions) the additional condition of continuity allows the finding of the lacking factors in stress function Φ .

If the radial distortion of any element of the ring is called u , and the tangential distortion of the same element v , then the values of $\widehat{rr}$, $\widehat{\theta\theta}$ and $\widehat{r\theta}$ stresses may be determined as functions of u and v as follows²⁾:

$$\left. \begin{aligned} \widehat{rr} &= (\lambda + 2\eta) \frac{\partial u}{\partial r} + \lambda \left(\frac{u}{r} + \frac{1}{r} \cdot \frac{\partial v}{\partial \theta} \right) \\ \widehat{\theta\theta} &= \lambda \frac{\partial u}{\partial r} + (\lambda + 2\eta) \left(\frac{u}{r} + \frac{1}{r} \cdot \frac{\partial v}{\partial \theta} \right) \\ \widehat{r\theta} &= \eta \left(\frac{\partial v}{\partial r} + \frac{1}{r} \cdot \frac{\partial u}{\partial \theta} - \frac{v}{r} \right) \end{aligned} \right\} \dots \dots \dots (15)$$

where $\eta \equiv G = \frac{E}{2(1+\mu)}$ is the modulus of elasticity in shear,

$\lambda = \frac{E\mu}{(1+\mu)(1-2\mu)}$ for a problem with two dimensional strain,

and $\lambda = \frac{\mu E}{1-\mu^2}$ for a problem with two dimensional stress.

After the substitution of values of $\widehat{rr}$, $\widehat{\theta\theta}$, $\widehat{r\theta}$ from (13a, 13b, 13c) into (15), the formulas for u and v will be obtained.

$$\left. \begin{aligned} u = & -\frac{a_0}{2\eta r} + \left(\frac{b_0}{\lambda + \eta} - \frac{c_0}{2\eta} \right) r + \frac{c_0}{\lambda + \eta} r \log r + \left[\frac{a_1}{4(\lambda + \eta)} + \frac{\lambda + 2\eta}{2\eta(\lambda + \eta)} \beta_1 \right] \theta \cdot \sin \theta + \\ & - \left[\frac{c_1}{4(\lambda + \eta)} + \frac{\lambda + 2\eta}{2\eta(\lambda + \eta)} \delta_1 \right] \theta \cdot \cos \theta + \left[\left(\frac{\lambda + 2\eta}{4\eta(\lambda + \eta)} a_1 + \frac{\beta_1}{2(\lambda + \eta)} \right) \log r + \right. \\ & \left. + \frac{\eta - \lambda}{2\eta(\lambda + \eta)} b_1 r^2 + \frac{a_1}{2\eta r^2} + \frac{1}{8\eta} \gamma_w \cdot r^2 + A \right] \cos \theta + \\ & + \left[\left(\frac{\lambda + 2\eta}{4\eta(\lambda + \eta)} c_1 + \frac{\delta_1}{2(\lambda + \eta)} \right) \log r + \frac{\eta - \lambda}{2\eta(\lambda + \eta)} d_1 r^2 + \frac{\gamma_1}{2\eta r^2} + B \right] \sin \theta + \\ & + \sum_{n=2}^{\infty} \left[-\frac{n}{2\eta} a_n r^{n-1} - \frac{n\lambda + (n-2)\eta}{2\eta(\lambda + \eta)} b_n r^{n+1} + \frac{n}{2\eta} a_n r^{-n-1} + \right. \\ & \left. + \frac{n\lambda + (n+2)\eta}{2\eta(\lambda + \eta)} \beta_n r^{-n+1} \right] \cos n\theta + \\ & + \sum_{n=2}^{\infty} \left[-\frac{n}{2\eta} c_n r^{n-1} - \frac{n\lambda + (n-2)\eta}{2\eta(\lambda + \eta)} d_n r^{n+1} + \frac{n\gamma_n}{2\eta} r^{-n-1} + \right. \\ & \left. + \frac{n\lambda + (n+2)\eta}{2\eta(\lambda + \eta)} \delta_n r^{-n+1} \right] \sin n\theta \end{aligned} \right\} (16)$$

¹⁾ A. Timpe. "Zeitschrift für Mathematik und Physik", 1905, page 374.

²⁾ A. E. H. Love. "Treatise on the Theory of Elasticity".

$$\begin{aligned}
 v = & Cr - \frac{\alpha_0}{2\eta r} + \frac{\lambda + 2\eta}{\eta(\lambda + \eta)} c_0 r\Theta + \left[\frac{a_1}{4(\lambda + \eta)} + \frac{\lambda + 2\eta}{2\eta(\lambda + \eta)} \beta_1 \right] \Theta \cos \Theta + \\
 & + \left[\frac{c_1}{4(\lambda + \eta)} + \frac{\lambda + 2\eta}{2\eta(\lambda + \eta)} \delta_1 \right] \Theta \sin \Theta + \left[\left(-\frac{\lambda + 2\eta}{4\eta(\lambda + \eta)} \alpha_1 - \frac{\beta_1}{2(\lambda + \eta)} \right) \log r + \right. \\
 & \left. - \frac{a_1}{4\eta} + \frac{3\lambda + 5\eta}{2\eta(\lambda + \eta)} b_1 r^2 + \frac{\alpha_1}{2\eta r^2} - \frac{3}{8\eta} \gamma_w r^2 - A \right] \sin \Theta + \\
 & + \left[\left(\frac{\lambda + 2\eta}{4\eta(\lambda + \eta)} c_1 + \frac{\delta_1}{2(\lambda + \eta)} \right) \log r + \frac{c_1}{4\eta} - \frac{3\lambda + 5\eta}{2\eta(\lambda + \eta)} d_1 r^2 - \frac{\gamma_1}{2\eta r^2} + B \right] \cos \Theta + \\
 & + \sum_{n=2}^{\infty} \left[\frac{n}{2\eta} a_n r^{n-1} + \frac{(n+4)(\lambda + 2\eta) + n\lambda}{4\eta(\lambda + \eta)} b_n r^{n+1} + \frac{n}{2\eta} a_n r^{-n-1} + \right. \\
 & \left. + \frac{(n-4)(\lambda + 2\eta) + n\lambda}{4\eta(\lambda + \eta)} \beta_n r^{-n+1} \right] \sin n\Theta + \\
 & + \sum_{n=2}^{\infty} \left[-\frac{n}{2\eta} c_n r^{n-1} - \frac{(n+4)(\lambda + 2\eta) + n\lambda}{4\eta(\lambda + \eta)} d_n r^{n+1} - \frac{n}{2\eta} \gamma_n r^{-n-1} + \right. \\
 & \left. - \frac{(n-4)(\lambda + 2\eta) + n\lambda}{4\eta(\lambda + \eta)} \delta_n r^{-n+1} \right] \cos n\Theta.
 \end{aligned} \tag{17}$$

It will be noticed, that the effect of body forces is expressed in formulas (16) and (17) as:

$$\begin{aligned}
 u_1 &= \frac{1}{8\eta} \gamma_w \cdot r^2 \cdot \cos \Theta + A \cos \Theta + B \sin \Theta \\
 v_1 &= -\frac{3}{8\eta} \gamma_w \cdot r^2 \cdot \sin \Theta - A \sin \Theta + B \cos \Theta
 \end{aligned}$$

Values of coefficients A , B , C , depend on the methods of supporting the ring.

As follows from (16) and (17), due to the condition of the continuity of distortion, the coefficients of $\Theta \cdot \sin \Theta$, $\Theta \cdot \cos \Theta$ and $r \cdot \Theta$ must be equal to zero. Which gives:

$$\left. \begin{aligned} \delta_1 &= -\frac{\eta}{2(\lambda + 2\eta)} c_1; & \beta_1 &= -\frac{\eta}{2(\lambda + 2\eta)} a_1; & c_0 &= 0; \end{aligned} \right\} \dots \tag{18}$$

All other coefficients in formulas (13a-b-c) for $n=0$ and $n=1$ now may be easily determined.

For $n=0$ there are relations:

$$\left. \begin{aligned} \frac{\alpha_0}{R_1^2} + 2b_0 &= A_0 \\ \frac{\alpha_0}{R_2^2} + 2b_0 &= A_0' \\ \frac{\alpha_0}{R_1^2} &= C_0 \\ \frac{\alpha_0}{R_2^2} &= C_0' \end{aligned} \right\} \tag{19}$$

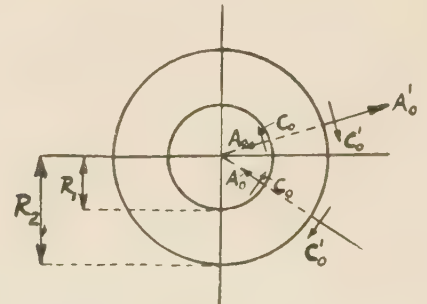


Fig. 4.

as follows from the last two equations: $C_0 \cdot R_1^2 = C_0' \cdot R_2^2$ which is in agreement with the conditions of equilibrium for, the sum of the moments about the center of the ring (Fig. 4) gives the relation:

$$\int_0^{2\pi} C_0' \cdot R_2^2 \cdot d\Theta = \int_0^{2\pi} C_0 \cdot R_1^2 \cdot d\Theta.$$

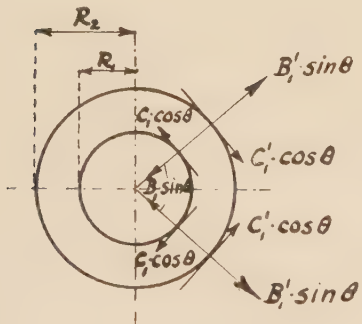
From relations (19) there are given values of a_0, b_0, α_0 .

For $n = 1$.

With the assumption that body forces are acting in the direction of the y axis only, the condition will be obtained that in the formulas (13a) and (13b) coefficients of $\cos \Theta$ — contain γ_w ; those of $\sin \Theta$ — are independent of γ_w ; in the formula (13c) it is the contrary. (If there are also body forces in the direction of the x axis, then all terms with sines and cosines of the single angle Θ will contain γ'_w or γ''_w in a similar manner). The following relations may be written:

$$\left. \begin{aligned} \frac{a_1 + \beta_1}{R_1} + 2 b_1 R_1 - \frac{2 \alpha_1}{R_1^3} + \frac{1}{2} \gamma_w \cdot R_1 &= A_1 \\ \frac{a_1 + \beta_1}{R_2} + 2 b_1 R_2 - \frac{2 \alpha_1}{R_2^3} + \frac{1}{2} \gamma_w \cdot R_2 &= A_1' \\ 2 b_1 R_1 - \frac{2 \alpha_1}{R_1^3} + \frac{\beta_1}{R_1} - \frac{1}{2} \gamma_w \cdot R_1 &= D_1 \\ 2 b_1 R_2 - \frac{2 \alpha_1}{R_2^3} + \frac{\beta_1}{R_2} - \frac{1}{2} \gamma_w \cdot R_2 &= D_1' \end{aligned} \right\} \dots \dots \dots (20)$$

For terms without γ_w these will be:



$$\left. \begin{aligned} \frac{c_1 + \delta_1}{R_1} + 2 d_1 R_1 - \frac{2 \gamma_1}{R_1^3} &= B_1 \\ \frac{c_1 + \delta_1}{R_2} + 2 d_1 R_2 - \frac{2 \gamma_1}{R_2^3} &= B_1' \\ - 2 d_1 R_1 + \frac{2 \gamma_1}{R_1^3} - \frac{\delta_1}{R_1} &= C_1 \\ - 2 d_1 R_2 + \frac{2 \gamma_1}{R_2^3} - \frac{\delta_1}{R_2} &= C_1' \end{aligned} \right\} \dots \dots \dots (21)$$

Fig. 5.

As follows from (21) $\sim R_1 (B_1 + C_1) = R_2 (B_1' + C_1')$, which may easily be explained from the conditions of equilibrium (Fig. 5), by taking the sum of the projections on the horizontal axis:

$$\left. \begin{aligned} 2 \int_0^{\pi} B_1' \sin^2 \Theta R_2 d\Theta + 2 \int_0^{\pi} C_1' \cos^2 \Theta R_2 d\Theta - 2 \int_0^{\pi} B_1 \sin^2 \Theta R_1 d\Theta + \\ - 2 \int_0^{\pi} C_1 \cos^2 \Theta R_1 d\Theta &= 0 \end{aligned} \right\} \dots \dots (22)$$

which gives: $R_1 (B_1 + C_1) = R_2 (B_1' + C_1')$.

In the same manner the relation will be obtained from (20) by taking the sum of the projections on the vertical axis:

$$R_1(A_1 - D_1) - \gamma_w R_1^2 = R_2(A_1' - D_1') - \gamma_w R_2^2 \quad . \quad . \quad . \quad . \quad . \quad (22a)$$

The value of c_1 may be found from (21)

$$c_1 = R_1(B_1 + C_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

From (21) may be found also values of coefficients d_1 and γ_1 ; thus all coefficients in equations (21) are now determined [δ_1 is known from (18)]. The procedure with equations (20) is the same.

In the discussion of equations (21) (without body forces), it is necessary to add, that for the case when the external loading on the inner and on the outer boundary of the ring is given in the form of two independent systems of forces which are in equilibrium, or arrange themselves in two couples in equilibrium, then an additional condition follows:

$$B_1 + C_1 = 0, \quad B_1' + C_1' = 0,$$

as may be seen from (22) by taking the sum of the projections on the horizontal axis. Then formula (23) gives $c_1 = 0$ and formula (18) gives $\delta_1 = 0$; thus for this case the solution of equations (21) is independent of the elastic properties of the material.

From the system of equations (20) containing body forces the value of a_1 is obtained:

$$a_1 = R_1(A_1 - D_1) - \gamma_w \cdot R_1^2$$

a_1 equals zero when both sides of equation (22a) are equal zero; then $\beta_1 = 0$ as follows from (18). The explanation of this is identical to that of the previous case if special equivalent loading is applied.

As follows from (20), after the transformation of the terms with γ_w on the right side, a new system of equations will be obtained where all internal body forces are neglected, but the external loading on the inner and on the outer boundary of the ring is increased. To explain this it may be assumed that the weight of the ring may be replaced by a certain external loading given by the following formulas:

$$(\widehat{rr})_{r=R_1} = -\frac{1}{2} \gamma_w \cdot R_1 \cdot \cos \theta$$

$$(\widehat{r\theta})_{r=R_1} = +\frac{1}{2} \gamma_w \cdot R_1 \cdot \sin \theta$$

$$(\widehat{rr})_{r=R_2} = -\frac{1}{2} \gamma_w \cdot R_2 \cdot \cos \theta$$

$$(\widehat{r\theta})_{r=R_2} = +\frac{1}{2} \gamma_w \cdot R_2 \cdot \sin \theta$$

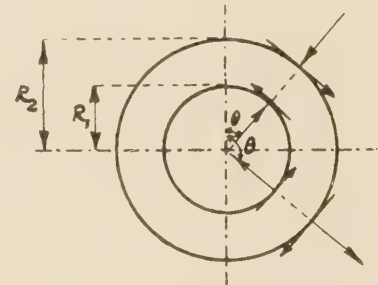


Fig. 6.

This will be called the "equivalent loading".

The values of σ_H and σ_v of horizontal and vertical components of "equivalent stresses" may now be determined.

On the inner boundary:

$$\sigma_v = -\frac{1}{2} \gamma_w \cdot R_1 \cdot \cos^2 \theta - \frac{1}{2} \gamma_w \cdot R_1 \cdot \sin^2 \theta = -\frac{1}{2} \gamma_w \cdot R_1$$

$$\sigma_H = -\frac{1}{2} \gamma_w \cdot R_1 \cdot \cos \theta \cdot \sin \theta + \frac{1}{2} \gamma_w \cdot \sin \theta \cdot \cos \theta = 0$$

$$\left. \begin{aligned} (\widehat{rr})_3 &= \left(-6 a_3 r - 4 b_3 r^3 - 12 \alpha_3 r^{-5} - 10 \beta_3 r^{-3} - \frac{\gamma_w}{2} r \right) \cos 3 \theta \\ (\widehat{\theta\theta})_3 &= \left(6 a_3 r + 20 b_3 r^3 + 12 \alpha_3 r^{-5} + 2 \beta_3 r^{-3} + \frac{\gamma_w}{2} r \right) \cos 3 \theta \\ (\widehat{r\theta})_3 &= \left(6 a_3 r + 12 b_3 r^3 - 12 \alpha_3 r^{-5} - 6 \beta_3 r^{-3} + \frac{\gamma_w}{2} r \right) \sin 3 \theta \end{aligned} \right\} \dots (26)$$

It is easy to notice that values of stresses as given in (26) are independent of γ_w . By taking terms with a_3 and γ_w :

$$6 a_3 r + \frac{\gamma_w}{2} r = 6 r \left(a_3 + \frac{\gamma_w}{12} \right) = 6 a_3' r$$

thus to find the coefficients a_3' , b_3 , α_3 , β_3 it is sufficient to take into account the terms with 3θ where γ_w is neglected.

III. Stresses in a circular ring under the action of body forces and a single concentrated force P .

In the previous considerations there is given the general solution of stress and strain problems of circular elastic rings under the action of body forces and external loading. Now an example will be taken of a circular ring under the action of it's own weight with a single concentrated force P acting along a vertical axis (Fig. 8):

$$P = \gamma_w \cdot \pi (R_2^2 - R_1^2)$$

As follows from equations (14) the values of external stresses may be given in the form of Fourier's series:

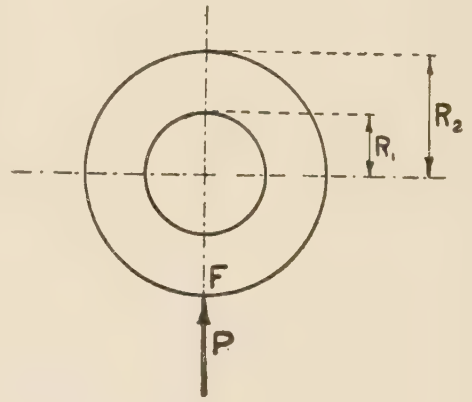


Fig. 8.

$$\left. \begin{aligned} (\widehat{rr})_{r=R_2} &= A_0 + \sum_{n=1}^{\infty} A_n \cos n \theta \\ (\widehat{r\theta})_{r=R_2} &= \sum_{n=1}^{\infty} D_n \sin n \theta \end{aligned} \right\} \dots (27)$$

Due to the symmetry of loading with respect to the vertical axis the values of $\widehat{rr}$ are expressed in a series of cosines, and $\widehat{r\theta}$ in a series of sines (for $\widehat{rr}$ has the same value for $\angle \theta$ and $\angle (-\theta)$, while $\widehat{r\theta}$ changes it's sign, when the sign of the angle is changed). Values of coefficients A_n and D_n will be determined by formulas:

$$\left. \begin{aligned} A_n &= \frac{2}{\pi} \int_0^\pi (\widehat{rr})_{r=R_2} \cos n\Theta \cdot d\Theta \\ D_n &= \frac{2}{\pi} \int_0^\pi (\widehat{r\Theta})_{r=R_2} \sin n\Theta \cdot d\Theta \end{aligned} \right\} \dots \dots \dots (28)$$

It may be assumed that force P consists of a uniformly distributed loading over the small arc of the angle 2ε . Thus the value of this loading for the angle $\pi - \varepsilon < \Theta < \pi + \varepsilon$

$$\text{is } (\widehat{rr})_{r=R_2} = -\frac{P}{2 R_2 \varepsilon} \quad \text{and} \quad (\widehat{r\Theta})_{r=R_2} = 0;$$

but for the angle $0 < \Theta < \pi - \varepsilon$ or $\pi + \varepsilon < \Theta < 2\pi$, $(\widehat{rr})_{r=R_2} = (\widehat{r\Theta})_{r=R_2} = 0$.

After the substitution of these values into (28):

$$A_n = -\frac{P}{\pi R_2 \varepsilon} \int_{\pi-\varepsilon}^\pi \cos n\Theta \cdot d\Theta = -\frac{P}{n \pi R_2 \varepsilon} \cos n\pi \cdot \sin(n\varepsilon)$$

but ε is approaching zero, thus $\lim_{\varepsilon \rightarrow 0} \frac{\sin(n\varepsilon)}{n\varepsilon} = 1$ and $A_n = -\frac{P}{\pi R_2} \cos n\pi$

In a similar manner:

$$A_0 = -\frac{P}{2 R_2 \pi \varepsilon} \int_{\pi-\varepsilon}^\pi d\Theta = -\frac{P}{2 R_2 \pi}$$

It is easy to see that $D_n = 0$, for $\widehat{r\Theta} = 0$ on the whole boundary. After the substitution of these values into (27) the final formulas of $\widehat{rr}$ and $\widehat{r\Theta}$ will be obtained:

$$\left. \begin{aligned} (\widehat{rr})_{r=R_2} &= -\frac{P}{\pi R_2} \left(\frac{1}{2} + \sum_{n=1}^{\infty} \cos n\pi \cdot \cos n\Theta \right) \\ (\widehat{r\Theta})_{r=R_2} &= 0. \end{aligned} \right\} \dots \dots \dots (27a)$$

To find the value of $(\widehat{rr})_{r=R_2}$ at any point of the outer boundary of the ring, formulas (27a) will be transformed. The contents of the brackets may be written in the form:

$$\frac{1}{2} + \sum_{n=1}^{\infty} \cos n(\pi - \Theta)$$

as follows from the identity, $\cos[n(\pi - \Theta)] = \cos n\pi \cdot \cos n\Theta$. Assuming that $\pi - \Theta$ is called α where α is measured from F the point of application of force P , the following series will be obtained¹⁾:

¹⁾ E. Kohl. "Ingenieur Archiv", 1930, page 211.
E. W. Hobson. Trigonometry.

$$\frac{1}{2} + \cos \alpha + \cos 2\alpha + \cos 3\alpha + \dots + \cos n\alpha.$$

The value of this sum, when $n \rightarrow \infty$ is given by the formula:

$$S_n = \frac{1}{2n} \left[\frac{\sin \frac{n}{2} \alpha}{\sin \frac{\alpha}{2}} \right]^2$$

As may easily be noticed $S_n \rightarrow \infty$ if $\alpha = 2k\pi$

and $S_n \rightarrow 0$ if $\alpha \neq 2k\pi$

It follows that $(\bar{r}r)_{r=R_2}$ is infinitely large at the point of application of the force P , but is equal to zero at every other point of the circumference of the ring. To find the values of internal stresses the formulas (13a-b-c) should be used. As follows from (27):

$$c_n = d_n = \epsilon_n = \delta_n = 0$$

and factors $a_n, b_n, \alpha_n, \beta_n$ for $n > 1$ will be determined after comparison of (13a-c) with (27):

$$\left. \begin{aligned} n(1-n)a_n R_1^{n-2} + (n+2-n^2)b_n R_1^n - n(1+n)\alpha_n R_1^{-n-2} - (n-2+n^2)\beta_n R_1^{-n} &= 0 \\ n(1-n)a_n R_2^{n-2} + (n+2-n^2)b_n R_2^n - n(1+n)\alpha_n R_2^{-n-2} - (n-2+n^2)\beta_n R_2^{-n} &= A_n \\ n(n-1)a_n R_1^{n-2} + n(n+1)b_n R_1^n - n(n+1)\alpha_n R_1^{-n-2} - n(n-1)\beta_n R_1^{-n} &= 0 \\ n(n-1)a_n R_2^{n-2} + n(n+1)b_n R_2^n - n(n+1)\alpha_n R_2^{-n-2} - n(n-1)\beta_n R_2^{-n} &= D_n \end{aligned} \right\} \quad (29)$$

Solution of this system gives the values of the factors considered, where the ratio $\frac{R_2}{R_1} = m$.

$$\left. \begin{aligned} a_n &= \frac{R_2^{-n+2} m^{2n-2} \{ (D_n + A_n)(n-1)(m^2 - m^{2n+2}) + (D_n - A_n)[(n^2-1)m^2 - n^2 + m^{2n+2}] \}}{2n(n-1)[2(n^2-1)m^{2n} - n^2 m^{2n+2} - n^2 m^{2n-2} + m^{4n} + 1]} \\ b_n &= \frac{R_2^{-n} m^{2n} [(D_n + A_n)(m^{2n} - m^2) + (n+1)(D_n - A_n)(1 - m^2)]}{2(n+1)[2(n^2-1)m^{2n} - n^2 m^{2n+2} - n^2 m^{2n-2} + m^{4n} + 1]} \\ \alpha_n &= \frac{R_2^{n+2} \{ n^2 (D_n + A_n) m^{2n-2} - [(n+2)D_n - nA_n] - (n+1)[nA_n + (n-2)D_n] m^{2n} \}}{2n(n+1)[2(n^2-1)m^{2n} - n^2 m^{2n+2} - n^2 m^{2n-2} + m^{4n} + 1]} \\ \beta_n &= \frac{R_2^n \{ (D_n - A_n) - m^{2n} (n-1)(D_n + A_n) - m^{2n+2} [2D_n - n(D_n + A_n)] \}}{2(n-1)[2(n^2-1)m^{2n} - n^2 m^{2n+2} - n^2 m^{2n-2} + m^{4n} + 1]} \end{aligned} \right\} \quad (30)$$

For $n=3$ the value of coefficient a_3' will be obtained from this formula; thus in formulas (13a-b-c) all terms with $\gamma_w \cdot \cos 3\theta$ and $\gamma_w \cdot \sin 3\theta$ should be omitted.

Values of factors $a_1, b_1, \alpha_1, \beta_1$ will be determined from (18) and (20):

$$\left. \begin{aligned} a_1 &= R_1(A_1 - D_1 - \gamma_w R_1) = R_2(A_1' - D_1' - \gamma_w R_2) \\ \beta_1 &= -\frac{\gamma_l}{2(\lambda + 2\gamma_l)} a_1 \\ \alpha_1 &= \frac{\beta_1}{2} \cdot \frac{R_1^2 \cdot R_2^2}{R_2^2 + R_1^2} + \frac{D_1' R_1 - D_1 R_2}{2} \cdot \frac{R_1^3 \cdot R_2^3}{R_2^4 - R_1^4} \\ b_1 &= \frac{\gamma_w}{4} - \frac{\beta_1}{2(R_2^2 + R_1^2)} + \frac{D_1' R_2^3 - D_1 R_1^3}{2(R_2^4 - R_1^4)} \end{aligned} \right\} \cdot \cdot \cdot \cdot (31)$$

or after the substitution $A_1 = D_1 = D_1' = 0$ and $A_1' = \frac{P}{\pi R_2}$

$$\left. \begin{aligned} a_1 &= -\gamma_w R_1^2 \\ \beta_1 &= \frac{\gamma_l}{2(\lambda + 2\gamma_l)} \cdot \gamma_w R_1^2 \\ \alpha_1 &= \frac{\gamma_l}{4(\lambda + 2\gamma_l)} \cdot \frac{\gamma_w R_1^4 R_2^2}{R_2^2 + R_1^2} \\ b_1 &= \frac{\gamma_w}{4} - \frac{\gamma_l}{4(\lambda + 2\gamma_l)} \cdot \frac{\gamma_w R_1^2}{R_2^2 + R_1^2} \end{aligned} \right\} \cdot \cdot \cdot \cdot (31a)$$

Values of the factors a_0, b_0, c_0, α_0 , will be determined from (18) and (19):

$$\left. \begin{aligned} c_0 &= 0 \\ \alpha_0 &= C_0 \cdot R_1^2 \\ a_0 &= (A_0 - A_0') \frac{R_1^2 R_2^2}{R_2^2 - R_1^2} \\ b_0 &= \frac{A_0' R_2^2 - A_0 R_1^2}{2(R_2^2 - R_1^2)} \end{aligned} \right\} \cdot \cdot \cdot \cdot (32)$$

or after the substitution $C_0 = C_0' = A_0 = 0$ and $A_0' = -\frac{P}{2\pi R_2}$

¹⁾ For the special case when $R_1 = 0$, that is when a disk problem instead of a ring problem is considered, there will be:

$$a_1 = \beta_1 = \alpha_1 = 0 \quad \text{and} \quad b_1 = \frac{\gamma_w}{4}$$

or the values of stresses are independent of the elastic properties of the material, for the "equivalent loading" with the force P forms a system of external stresses which are in equilibrium.

$$\left. \begin{aligned} c_0 &= \alpha_0 = 0 \\ a_0 &= \frac{P}{2\pi} \cdot \frac{R_1^2 R_2}{R_2^2 - R_1^2} \\ b_0 &= -\frac{P}{4\pi} \cdot \frac{R_2}{R_2^2 - R_1^2} \end{aligned} \right\} \dots \dots \dots (32a)$$

The same values of factors as given in (32a) may be obtained from Lamé's formula for rings with a uniform external loading.

When all unknown coefficients in the formulas (13a-b-c) are determined, the values of internal stresses $\widehat{rr}$, $\widehat{\theta\theta}$, $\widehat{r\theta}$, will be found by the summing up of the corresponding terms of these formulas. But there are many difficulties in the case when concentrated forces are applied on the boundaries of the ring. As follows from formula (28), the summing up of

Fourier's series should be done for an infinite number of terms $\sum_{n=1}^{\infty}$, for these series are

not rapidly convergent. Thus the method of summing up of these terms will now be considered. Here it should be noticed that to any value of $n > 1$ there corresponds:

$$(\widehat{rr})'_{r=R_2} = A_n \cos n\theta; \quad (\widehat{r\theta})'_{r=R_2} = D_n \sin n\theta$$

which form a system of stresses independent of body forces, and give a complete condition of equilibrium on the external boundary of the ring. Thus for any variation of the length of the radius of the unloaded circumference of the ring the value of P may be taken as a constant in the formulas for A_n and D_n (where $n > 1$). As a special case $R_1 = 0$ may be taken also; then instead of a ring, a disk with the same external circumference is considered.

Value of $m = \frac{R_2}{R_1} = \infty$ and values of coefficients $a_n, b_n, \alpha_n, \beta_n$, will be obtained from formulas (30) after the substitution of $m = \infty$.

$$\left. \begin{aligned} a_n &= \frac{2 D_n - n(D_n + A_n)}{2 n(n-1)} R_2^{-n+2} \\ b_n &= \frac{D_n + A_n}{2(n+1)} R_2^{-n} \\ \alpha_n &= \beta_n = 0 \end{aligned} \right\} \dots \dots \dots (33)$$

In the problem considered — $D_n = 0$, $A_n = -\frac{P}{\pi R_2} \cos n\pi$;

$$\text{thus} \quad a_n = \frac{P \cdot \cos(n\pi)}{2(n-1)\pi} R_2^{-n+1}, \quad b_n = -\frac{P \cdot \cos(n\pi)}{2(n+1)\pi} R_2^{-n-1}$$

Values of internal stresses corresponding to $n > 1$ (for the disk) may be obtained from (13a-b-c) as $(\widehat{rr})'$, $(\widehat{\theta\theta})'$, $(\widehat{r\theta})'$:

¹⁾ For the special case when $R_1 = 0$, the values of the factors are: $c_0 = a_0 = \alpha_0 = 0$ and $b_0 = -\frac{P}{4\pi R_2}$;

and the corresponding values of stresses: $\widehat{r\theta} = 0$, $\widehat{rr} = \widehat{\theta\theta} = -\frac{P}{2\pi R_2}$ which agree with the conditions of the external loading.

$$\left. \begin{aligned} (\widehat{rr})' &= \left(-\frac{P}{2\pi R_2} \right) \sum_{n=2}^{\infty} \left\{ \left[\left(\frac{R_2}{r} \right)^2 - 1 \right] n \left(\frac{r}{R_2} \right)^n + 2 \left(\frac{r}{R_2} \right)^n \right\} \cos n\pi \cdot \cos n\Theta \\ (\widehat{\Theta\Theta})' &= \frac{P}{2\pi R_2} \sum_{n=2}^{\infty} \left\{ \left[\left(\frac{R_2}{r} \right)^2 - 1 \right] n \left(\frac{r}{R_2} \right)^n - 2 \left(\frac{r}{R_2} \right)^n \right\} \cos n\pi \cdot \cos n\Theta \\ (\widehat{r\Theta})' &= \frac{P}{2\pi R_2} \sum_{n=2}^{\infty} \left[\left(\frac{R_2}{r} \right)^2 - 1 \right] n \left(\frac{r}{R_2} \right)^n \cos n\pi \cdot \sin n\Theta \end{aligned} \right\} \quad . \quad . \quad (34)$$

The summing up of these three types of series should be made to obtain the values of internal stresses. Assuming $\frac{r}{R_2} = x$ where $x < 1$

$$\left. \begin{aligned} \cos(n\pi) \cdot \cos(n\Theta) &= \cos(n\alpha) \\ \cos(n\pi) \cdot \sin(n\Theta) &= -\sin(n\alpha) \end{aligned} \right\} \quad \text{where } \alpha = \pi - \Theta$$

The sum of the following series should be determined:

$$\begin{aligned} n \cdot x^n \cdot \cos(n\alpha) \dots \dots \dots S_n' &= (1 + x \cos \alpha + 2x^2 \cos 2\alpha + \dots \dots \dots) \\ x^n \cdot \cos(n\alpha) \dots \dots \dots S_n'' &= (1 + x \cos \alpha + x^2 \cos 2\alpha + \dots \dots \dots) \\ n \cdot x^n \cdot \sin(n\alpha) \dots \dots \dots S_n''' &= (1 + x \sin \alpha + 2x^2 \sin 2\alpha + \dots \dots \dots) \end{aligned}$$

To find the values of S_n' , S_n'' , S_n''' of these Recurring Series respectively the relevant scale of relations of each series will be used. It may easily be found that for the series S_n'' this scale will be $(1 - 2x \cdot \cos \alpha + x^2)$, because:

$$(1 - 2x \cdot \cos \alpha + x^2) S_n'' = 1 - x \cdot \cos \alpha - x^{n+1} \cdot \cos(n+1)\alpha + x^{n+2} \cdot \cos(n\alpha) \quad (35)$$

Thus the limit of S_n'' with $n \rightarrow \infty$ (for $x < 1$) may be given by the formula:

$$S_{n \rightarrow \infty}'' = \frac{1 - x \cdot \cos \alpha}{1 - 2x \cdot \cos \alpha + x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35a)$$

In a similar manner this may be done for S_n' :

$$S_n'(1 - 2x \cdot \cos \alpha + x^2) = x^2 - 2x \cdot \cos \alpha + S_n'' - x^2 \cdot S_{n-1}'' - n \cdot x^{n+1} \cdot \cos(n+1)\alpha + nx^{n+2} \cdot \cos(n\alpha)$$

or after the substitution of the values of S_n'' and S_{n-1}'' as found from formula (35), this will read:

$$\left. \begin{aligned} S_n'(1 - 2x \cos \alpha + x^2) &= \frac{x^4 - 3x^3 \cos \alpha + 4x^2 \cos^2 \alpha - 3x \cos \alpha + 1}{1 - 2x \cos \alpha + x^2} + \\ &+ \frac{-x^{n+1} \cos(n+1)\alpha + 2x^{n+2} \cos n\alpha - x^{n+3} \cos(n-1)\alpha}{1 - 2x \cos \alpha + x^2} + \\ &+ nx^{n+2} \cos n\alpha - nx^{n+1} \cos(n+1)\alpha \end{aligned} \right\} \quad . \quad . \quad (36)$$

The limit of S_n' will be found when $n \rightarrow \infty$:

$$S_n' \rightarrow \frac{x^4 - 3x^3 \cos \alpha + 4x^2 \cos^2 \alpha - 3x \cos \alpha + 1}{(1 - 2x \cos \alpha + x^2)^2} \quad . \quad . \quad . \quad (36a)$$

To find the value of S_n''' the same factor will be applied¹⁾; then:

$$\begin{aligned} S_n'''(1 - 2x \cos \alpha + x^2) = & \frac{x^4 - 4x^3 \cos \alpha - x^3 \sin \alpha + 2x^2 + 4x^2 \cos^2 \alpha - 4x \cos \alpha + x \sin \alpha + 1}{1 - 2x \cos \alpha + x^2} + \\ & + \frac{-x^{n+1} \sin(n+1)\alpha + 2x^{n+2} \sin n\alpha - x^{n+3} \sin(n-1)\alpha}{1 - 2x \cos \alpha + x^2} + \\ & + nx^{n+2} \sin n\alpha - nx^{n+1} \sin(n-1)\alpha \quad . \quad . \quad . \quad (37) \end{aligned}$$

The limit of S_n''' is determined when $n \rightarrow \infty$:

$$S_n''' = \frac{x^4 - 4x^3 \cos \alpha - x^3 \sin \alpha + 2x^2 + 4x^2 \cos^2 \alpha - 4x \cos \alpha + x \sin \alpha + 1}{(1 - 2x \cos \alpha + x^2)^2} \quad . \quad . \quad (37a)$$

Now, when values of the sums S_n' , S_n'' , S_n''' are determined, it is easy to find the values of stresses $(\widehat{rr})'$, $(\widehat{\theta\theta})'$, $(\widehat{r\theta})'$ as given in formulas (34) for the disk:

$$\begin{aligned} (\widehat{rr})' = & \left(-\frac{P}{2\pi R_2} \right) \left\{ \left(\frac{R_2}{r} \right)^2 - 1 \right\} \left(\frac{x^4 - 3x^3 \cos \alpha + 4x^2 \cos^2 \alpha - 3x \cos \alpha + 1}{(1 - 2x \cos \alpha + x^2)^2} - x \cos \alpha - 1 \right) + \\ & + 2 \left(\frac{1 - x \cos \alpha}{1 - 2x \cos \alpha + x^2} - x \cos \alpha - 1 \right) \end{aligned}$$

or after the transformations:

$$\begin{aligned} (\widehat{rr})' = & \left(-\frac{P}{2\pi R_2} \right) \left\{ \frac{(1-x^2)}{x} \left[\frac{(x^2+1) \cos \alpha - 2x}{(1-2x \cos \alpha + x^2)^2} - \cos \alpha \right] + \right. \\ & \left. + 2x \left[\frac{\cos \alpha - x}{1 - 2x \cos \alpha + x^2} - \cos \alpha \right] \right\} \quad . \quad . \quad . \quad (38) \end{aligned}$$

In a similar manner the values of $(\widehat{\theta\theta})'$ and $(\widehat{r\theta})'$ may be determined:

$$\begin{aligned} (\widehat{\theta\theta})' = & \frac{P}{2\pi R_2} \left\{ \frac{1-x^2}{x} \left[\frac{(x^2+1) \cos \alpha - 2x}{(1-2x \cos \alpha + x^2)^2} - \cos \alpha \right] + \right. \\ & \left. - 2x \left[\frac{\cos \alpha - x}{1 - 2x \cos \alpha + x^2} - \cos \alpha \right] \right\} \quad . \quad . \quad . \quad (39) \end{aligned}$$

¹⁾ The value of sum $S_n = 1 + x \cdot \sin \alpha + x^2 \cdot \sin 2\alpha + x^3 \cdot \sin 3\alpha + \dots + x^n \cdot \sin(n\alpha)$ is given by formula:

$$S_n = \frac{1 + x^2 + x \cdot \sin \alpha - 2x \cdot \cos \alpha - x^{n+1} \cdot \sin(n+1)\alpha + x^{n+2} \cdot \sin(n\alpha)}{1 - 2x \cdot \cos \alpha + x^2}$$

$$(\widehat{r\theta})' = \left(-\frac{P}{2\pi R_2} \right) \left[\left(\frac{R_2}{r} \right)^2 + 1 \right] - 1 \left[\frac{x^4 - 4x^3 \cos \alpha - x^3 \sin \alpha + 2x^2 + 4x^2 \cos^2 \alpha - 4x \cos \alpha + x \sin \alpha + 1}{(1 - 2x \cos \alpha + x^2)^2} - x \sin \alpha - 1 \right]$$

or after the transformations:

$$(\widehat{r\theta})' = \frac{P}{2\pi R_2} \cdot \frac{1 - x^2}{x} \sin \alpha \left[\frac{x^2 - 1}{(1 - 2x \cos \alpha + x^2)^2} + 1 \right] \quad . \quad . \quad . \quad . \quad (40)$$

Thus the values of the internal stresses of the disk may be determined if the values of $(\widehat{rr})'$, $(\widehat{\theta\theta})'$, $(\widehat{r\theta})'$ are added to the corresponding terms in the formulas (13a-b-c) where $n = 1$, and $n = 0$ (without trigonometrical functions). The factors of these terms are given in formulas (31) and (32) with the substitution of $R_1 = 0$.

To solve the ring problem, the following method may be applied. In the formulas (13a-b-c) the terms with $n = 1$ and $n = 0$ are known for the ring; all other terms

expressed as $\sum_{n=2}^{\infty}$ are solved for the disk with the same external boundary. From the formulas

of $(\widehat{rr})'$ and $(\widehat{r\theta})'$ the value of these stresses on the boundary with R_1 as a radius may be found; thus when the inner part of the disk is removed and $(\widehat{rr})'$ and $(\widehat{r\theta})'$ stresses with the opposite sign are applied on the internal boundary of the ring, all conditions of the ring problem are obtained. It remains only to find the stresses due to continuous loading $-(\widehat{rr})'$, $-(\widehat{r\theta})'$ applied on the internal boundary and this may easily be done by the application of a very rapidly convergent Fourier's series. Final formulas of stresses will be obtained as the summing up of terms with $n = 1$ and $n = 0$ from (13a-b-c) with stresses from (38), (39), (40) and stresses due to $-(\widehat{rr})'$, $-(\widehat{r\theta})'$ loading, as given above.

As an example, a ring may be taken with the ratio $\frac{R_2}{R_1} = 3$. Thus to find the values of stresses on the internal boundary, formulas (38 and 40) must be applied with $x = \frac{1}{3}$ and the stresses obtained should be taken with the opposite sign. Then:

$$(\widehat{rr})' = \frac{P}{3\pi R_2} \left\{ \frac{1}{5 - 3 \cos \alpha} \left[\frac{90 \cos \alpha - 54}{5 - 3 \cos \alpha} + 1,5 (3 \cos \alpha - 1) \right] - 5 \cos \alpha \right\} \quad . \quad . \quad . \quad . \quad (38a)$$

$$(\widehat{r\theta})' = \frac{4P}{3\pi R_2} \left[\frac{18 \sin \alpha}{(5 - 3 \cos \alpha)^2} - \sin \alpha \right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40a)$$

Development of formulas (38a) and (40a) into Fourier's series.

The following notations are applied:

$$M = \frac{1}{5 - 3 \cos \alpha} \left[\frac{90 \cos \alpha - 54}{5 - 3 \cos \alpha} + 1,5 (3 \cos \alpha - 1) \right]$$

$$N = \frac{18 \sin \alpha}{(5 - 3 \cos \alpha)^2}$$

Table of values of M and N as functions of the angle α on the boundary of the ring.

α	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°	100°	110°	120°	130°	140°	150°	160°	170°	180°
M	10,5	9,7	7,7	5,15	2,77	0,86	-0,52	-1,46	-2,07	2,46	-2,70	-2,84	-2,92	-2,95	2,99	3,00	-3,00	-3,00	3,00
N	0	0,75	1,29	1,56	1,585	1,46	1,27	1,07	0,88	0,72	0,581	0,466	0,369	0,287	0,217	0,156	0,101	0,049	0

Values of M may be expressed here in a series of cosines and values of N in a series of sines as for the case of formulas (27).

$$M = A_0 + A_1 \cos \alpha + A_2 \cos 2\alpha + A_3 \cos 3\alpha + A_4 \cos 4\alpha + A_5 \cos 5\alpha + A_6 \cos 6\alpha + \left. \begin{aligned} &+ A_7 \cos 7\alpha + A_8 \cos 8\alpha + A_9 \cos 9\alpha \end{aligned} \right\} \quad (41)$$

$$N = D_1 \sin \alpha + D_2 \sin 2\alpha + D_3 \sin 3\alpha + D_4 \sin 4\alpha + D_5 \sin 5\alpha + D_6 \sin 6\alpha + \left. \begin{aligned} &+ D_7 \sin 7\alpha + D_8 \sin 8\alpha + D_9 \sin 9\alpha \end{aligned} \right\} \quad (42)$$

Values of A_n and B_n with sufficient accuracy are given by formulas:

$$A_0 = \frac{1}{36} \sum_{36} (M)_\alpha; \quad A_n = \frac{1}{18} \sum_{36} (M)_\alpha \cos(n\alpha); \quad B_n = \frac{1}{18} \sum_{36} (N)_\alpha \sin(n\alpha).$$

A_0	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9
0	$\frac{90,00}{18}$	$\frac{54,00}{18}$	$\frac{26,02}{18}$	$\frac{11,31}{18}$	$\frac{4,65}{18}$	$\frac{1,84}{18}$	$\frac{0,625}{18}$	$\frac{0,25}{18}$	$\frac{0,08}{18}$

D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9
$\frac{9,00}{9}$	$\frac{6,00}{9}$	$\frac{3,00}{9}$	$\frac{1,34}{9}$	$\frac{0,555}{9}$	$\frac{0,218}{9}$	$\frac{0,085}{9}$	$\frac{0,035}{9}$	$\frac{0,014}{9}$

After the substitution of the values of M and N into (38a) and (40a), formulas of $(\widehat{rr})'$ and $(\widehat{r\theta})'$ will be obtained:

$$\begin{aligned} (\widehat{rr})' &= \frac{P}{54 \pi R_2} (54,00 \cos 2\alpha + 26,02 \cos 3\alpha + 11,31 \cos 4\alpha + 4,65 \cos 5\alpha + \\ &\quad + 1,84 \cos 6\alpha + 0,625 \cos 7\alpha + 0,25 \cos 8\alpha + 0,08 \cos 9\alpha) \\ (\widehat{r\theta})' &= \frac{4 P}{27 \pi R_2} (6,00 \sin 2\alpha + 3,00 \sin 3\alpha + 1,34 \sin 4\alpha + 0,555 \sin 5\alpha + \\ &\quad + 0,218 \sin 6\alpha + 0,085 \sin 7\alpha + 0,035 \sin 8\alpha + 0,014 \sin 9\alpha) \end{aligned} \quad (43)$$

or after the substitution $\alpha = \pi - \Theta$

$$\left. \begin{aligned} (\overline{r r})' &= \frac{P}{54 \pi R_2} (54 \cos 2 \Theta - 26,02 \cos 3 \Theta + 11,31 \cos 4 \Theta - 4,65 \cos 5 \Theta + \\ &\quad + 1,84 \cos 6 \Theta - 0,625 \cos 7 \Theta + 0,25 \cos 8 \Theta - 0,08 \cos 9 \Theta) \\ (\overline{r \Theta})' &= \frac{P}{54 \pi R_2} (-48 \sin 2 \Theta + 24 \sin 3 \Theta - 10,72 \sin 4 \Theta + 4,44 \sin 5 \Theta + \\ &\quad - 1,744 \sin 6 \Theta + 0,68 \sin 7 \Theta - 0,28 \sin 8 \Theta + 0,112 \sin 9 \Theta) \end{aligned} \right\} \quad (43a)$$

To find the values of internal stresses of the ring under the action of loading $(\overline{r r})'$ and $(\overline{r \Theta})'$ on the inner boundary, formulas (13a-b-c) will be used with factors a_n , b_n , α_n , β_n , for $n > 1$ given by formulas (30) where R_2 must be changed to $R_1 = \frac{R_2}{m}$ and m to m^{-1} . Then:

$$\left. \begin{aligned} a_n &= \frac{m^{n-2} \cdot R_2^{-n+2} \{ (D_n + A_n)(n-1)(m^{2n} - 1) + (D_n - A_n)[(n^2 - 1)m^{2n} - n^2 \cdot m^{2n+2} + 1] \}}{2n(n-1)[2(n^2 - 1)m^{2n} - n^2 m^{2n+2} - n^2 m^{2n-2} + m^{4n} + 1]} \\ b_n &= \frac{m^n \cdot R_2^{-n} [(D_n + A_n)(1 - m^{2n-2}) + (n+1)(D_n - A_n)(m^{2n} - m^{2n-2})]}{2(n+1)[2(n^2 - 1)m^{2n} - n^2 m^{2n+2} - n^2 m^{2n-2} + m^{4n} + 1]} \\ \alpha_n &= \frac{m^{n-2} \cdot R_2^{n+2} \{ n^2(D_n + A_n)m^2 - [(n+2)D_n - nA_n]m^{2n} - (n+1)[nA_n + (n-2)D_n] \}}{2n(n+1)[2(n^2 - 1)m^{2n} - n^2 m^{2n+2} - n^2 m^{2n-2} + m^{4n} + 1]} \\ \beta_n &= \frac{m^n \cdot R_2^n \{ (D_n - A_n)m^{2n} - (n-1)(D_n + A_n) - m^{-2}[2D_n - n(D_n + A_n)] \}}{2(n-1)[2(n^2 - 1)m^{2n} - n^2 m^{2n+2} - n^2 m^{2n-2} + m^{4n} + 1]} \end{aligned} \right\} \quad (44)$$

Values of A_n and D_n are given as factors in formulas (43a) of $(\overline{r r})'$ and $(\overline{r \Theta})'$ and $m = 3$ for the present case. After the substitution of these values into (44) there will be obtained a_n , b_n , α_n , β_n for the different values of n .

n	2	3	4	5
a_n	$16,68 \frac{P}{54 \pi R_2}$	$-1,38 \frac{P}{54 \pi R_2^2}$	$0,166 \frac{P}{54 \pi R_2^3}$	$0,021 \frac{P}{54 \pi R_2^4}$
b_n	$-8,095 \frac{P}{54 \pi R_2^3}$	$0,905 \frac{P}{54 \pi R_2^4}$	$-0,123 \frac{P}{54 \pi R_2^5}$	$0,0167 \frac{P}{54 \pi R_2^6}$
α_n	$0,492 \frac{P R_2^3}{54 \pi}$	$-0,037 \frac{P R_2^4}{54 \pi}$	$0,0038 \frac{P R_2^5}{54 \pi}$	$-0,00041 \frac{P R_2^6}{54 \pi}$
β_n	$-9,08 \frac{P R_2}{54 \pi}$	$0,51 \frac{P R_2^2}{54 \pi}$	$-0,046 \frac{P R_2^3}{54 \pi}$	$0,0047 \frac{P R_2^4}{54 \pi}$

n	6	7	8	9
a_n	$0,00263 \frac{P}{54 \pi R_2^5}$	$-0,00031 \frac{P}{54 \pi R_2^6}$	$0,000041 \frac{P}{54 \pi R_2^7}$	$-0,0000049 \frac{P}{54 \pi R_2^8}$
b_n	$-0,00219 \frac{P}{54 \pi R_2^7}$	$0,000265 \frac{P}{54 \pi R_2^8}$	$-0,000036 \frac{P}{54 \pi R_2^9}$	$0,0000043 \frac{P}{54 \pi R_2^{10}}$
α_n	$0,000045 \frac{P R_2^7}{54 \pi}$	$-0,0000048 \frac{P R_2^8}{54 \pi}$	$0,00000056 \frac{P R_2^9}{54 \pi}$	$-0,000000061 \frac{P R_2^{10}}{54 \pi}$
β_n	$-0,00049 \frac{P R_2^5}{54 \pi}$	$0,00005 \frac{P R_2^6}{54 \pi}$	$-0,0000058 \frac{P R_2^7}{54 \pi}$	$0,00000061 \frac{P R_2^8}{54 \pi}$

After the substitution of the values of a_n , b_n , α_n , β_n into (13a-b-c), the corresponding values of internal stresses will be obtained.

Final values of the stresses in the ring $(\bar{r}r)$, $(\bar{\theta}\theta)$, $(\bar{r}\theta)$ may be found as a sum of the following terms: terms without trigonometrical functions with coefficients given by formulas (32a), terms with single angle θ — by formulas (31a), terms representing stresses in the disk given by (38), (39) and (40) and stresses found from (13a-b-c) with coefficients given in the last table. As a result of this summing up the following formulas will be obtained (after the substitution $x = \frac{r}{R_2}$, $R_1 = \frac{R_2}{3}$)

$$\begin{aligned}
 \bar{r}r = & \frac{P}{16 \pi r} \left(\frac{R_2}{r} - 9 \frac{r}{R_2} \right) + \left[1 - \frac{\eta}{20(\lambda + 2\eta)} + \left(\frac{\eta}{2(\lambda + 2\eta)} - 1 \right) \frac{R_2^2}{9r^2} + \right. \\
 & - \frac{\eta}{180(\lambda + 2\eta)} \cdot \frac{R_2^4}{r^4} \left. \right] r \gamma_w \cos \theta + \frac{P}{2\pi R_2} \left\{ \frac{1 - \frac{r^2}{R_2^2}}{\frac{r}{R_2}} \left[\frac{\left(\frac{r^2}{R_2^2} + 1 \right) \cos \theta + 2 \frac{r}{R_2}}{\left(1 + 2 \frac{r}{R_2} \cos \theta + \frac{r^2}{R_2^2} \right)^2} - \cos \theta \right] + \right. \\
 & + 2 \frac{r}{R_2} \left[\frac{\cos \theta + \frac{r}{R_2}}{1 + 2 \frac{r}{R_2} \cos \theta + \frac{r^2}{R_2^2}} - \cos \theta \right] \left. \right\} + \frac{P}{54 \pi R_2} \left[\left(-33,36 - 2,952 \frac{R_2^4}{r^4} + 36,32 \frac{R_2^2}{r^2} \right) \cos 2\theta + \right. \\
 & + \left(8,28 \frac{r}{R_2} - 3,62 \frac{r^3}{R_2^3} + 0,444 \frac{R_2^5}{r^5} - 5,10 \frac{R_2^3}{r^3} \right) \cos 3\theta + \left(-1,992 \frac{r^2}{R_2^2} + 1,23 \frac{r^4}{R_2^4} + \right. \\
 & - 0,076 \frac{R_2^6}{r^6} + 0,830 \frac{R_2^4}{r^4} \left. \right) \cos 4\theta + \left(0,42 \frac{r^3}{R_2^3} - 0,3006 \frac{r^5}{R_2^5} + 0,0123 \frac{R_2^7}{r^7} - 0,1316 \frac{R_2^5}{r^5} \right) \cos 5\theta + \\
 & + \left(-0,079 \frac{r^4}{R_2^4} + 0,0613 \frac{r^6}{R_2^6} - 0,00189 \frac{R_2^8}{r^8} + 0,0196 \frac{R_2^6}{r^6} \right) \cos 6\theta + \left(0,01302 \frac{r^5}{R_2^5} - 0,0106 \frac{r^7}{R_2^7} + \right. \\
 & + 0,00027 \frac{R_2^9}{r^9} - 0,0027 \frac{R_2^7}{r^7} \left. \right) \cos 7\theta + \left(-0,0023 \frac{r^6}{R_2^6} + 0,00194 \frac{r^8}{R_2^8} - 0,000040 \frac{R_2^{10}}{r^{10}} + \right. \\
 & + 0,000406 \frac{R_2^8}{r^8} \left. \right) \cos 8\theta + \left(0,00035 \frac{r^7}{R_2^7} - 0,00030 \frac{r^9}{R_2^9} + 0,0000055 \frac{R_2^{11}}{r^{11}} - 0,000054 \frac{R_2^9}{r^9} \right) \cos 9\theta \left. \right]
 \end{aligned} \tag{45}$$

$$\begin{aligned}
 \Theta\Theta = & - \frac{P}{16 \pi r} \left(\frac{R_2}{r} + 9 \frac{r}{R_2} \right) + \left[1 - \frac{3 \gamma_1}{20 (\lambda + 2 \gamma_1)} + \frac{\gamma_1}{180 (\lambda + 2 \gamma_1)} \cdot \frac{R_2^4}{r^4} + \right. \\
 & \left. + \frac{\gamma_1}{18 (\lambda + 2 \gamma_1)} \cdot \frac{R_2^2}{r^2} \right] r \gamma_w \cos \Theta + \\
 & - \frac{P}{2 \pi R_2} \left\{ \frac{1 - \frac{r^2}{R_2^2}}{\frac{r}{R_2}} \left[\frac{\left(\frac{r^2}{R_2^2} + 1 \right) \cos \Theta + 2 \frac{r}{R_2}}{\left(1 + 2 \frac{r}{R_2} \cos \Theta + \frac{r^2}{R_2^2} \right)^2} - \cos \Theta \right] + \right. \\
 & \left. - 2 \frac{r}{R_2} \left[\frac{\cos \Theta + \frac{r}{R_2}}{1 + 2 \frac{r}{R_2} \cos \Theta + \frac{r^2}{R_2^2}} - \cos \Theta \right] \right\} + \\
 & + \frac{P}{54 \pi R_2} \left[\left(33,36 + 2,952 \frac{R_2^4}{r^4} - 97,14 \frac{r^2}{R_2^2} \right) \cos 2 \Theta + \right. \\
 & + \left(- 8,28 \frac{r}{R_2} + 18,10 \frac{r^3}{R_2^3} - 0,444 \frac{R_2^5}{r^5} + 1,02 \frac{R_2^3}{r^3} \right) \cos 3 \Theta + \\
 & + \left(1,992 \frac{r^2}{R_2^2} - 3,69 \frac{r^4}{R_2^4} + 0,076 \frac{R_2^6}{r^6} - 0,276 \frac{R_2^4}{r^4} \right) \cos 4 \Theta + \\
 & + \left(- 0,42 \frac{r^3}{R_2^3} + 0,7014 \frac{r^5}{R_2^5} - 0,0123 \frac{R_2^7}{r^7} + 0,0564 \frac{R_2^5}{r^5} \right) \cos 5 \Theta + \\
 & + \left(0,079 \frac{r^4}{R_2^4} - 0,1226 \frac{r^6}{R_2^6} + 0,00189 \frac{R_2^8}{r^8} - 0,0098 \frac{R_2^6}{r^6} \right) \cos 6 \Theta + \\
 & + \left(- 0,01302 \frac{r^5}{R_2^5} + 0,01484 \frac{r^7}{R_2^7} - 0,00027 \frac{R_2^9}{r^9} + 0,0015 \frac{R_2^7}{r^7} \right) \cos 7 \Theta + \\
 & + \left(0,0023 \frac{r^6}{R_2^6} - 0,00324 \frac{r^8}{R_2^8} + 0,000040 \frac{R_2^{10}}{r^{10}} - 0,00024 \frac{R_2^8}{r^8} \right) \cos 8 \Theta + \\
 & + \left(- 0,00035 \frac{r^7}{R_2^7} + 0,000473 \frac{r^9}{R_2^9} - 0,0000055 \frac{R_2^{11}}{r^{11}} + 0,000034 \frac{R_2^9}{r^9} \right) \cos 9 \Theta \Big]
 \end{aligned} \tag{46}$$

$$\begin{aligned}
 \widehat{r\Theta} = & \left[-\frac{\gamma_l}{20(\lambda + 2\gamma_l)} - \frac{\gamma_l}{180(\lambda + 2\gamma_l)} \cdot \frac{R_2^4}{r^4} + \right. \\
 & \left. + \frac{\gamma_l}{18(\lambda + 2\gamma_l)} \cdot \frac{R_2^2}{r^2} \right] r \gamma_w \sin \Theta + \\
 & + \frac{P}{2\pi R_2} \cdot \frac{1 - \frac{r^2}{R_2^2}}{\frac{r}{R_2}} \left[\frac{\frac{r^2}{R_2^2} - 1}{\left(1 + 2\frac{r}{R_2} \cos \Theta + \frac{r^2}{R_2^2}\right)^2} + 1 \right] \sin \Theta + \\
 & + \frac{P}{54\pi R_2} \left[\left(33,36 - 48,57 \frac{r^2}{R_2^2} - 2,952 \frac{R_2^4}{r^4} + 18,16 \frac{R_2^2}{r^2} \right) \sin 2\Theta + \right. \\
 & + \left(-8,28 \frac{r}{R_2} + 10,86 \frac{r^3}{R_2^3} + 0,444 \frac{R_2^5}{r^5} - 3,06 \frac{R_2^3}{r^3} \right) \sin 3\Theta + \\
 & + \left(1,992 \frac{r^2}{R_2^2} - 2,46 \frac{r^4}{R_2^4} - 0,076 \frac{R_2^6}{r^6} + 0,552 \frac{R_2^4}{r^4} \right) \sin 4\Theta + \\
 & + \left(-0,42 \frac{r^3}{R_2^3} + 0,501 \frac{r^5}{R_2^5} + 0,0123 \frac{R_2^7}{r^7} - 0,094 \frac{R_2^5}{r^5} \right) \sin 5\Theta + \\
 & + \left(0,079 \frac{r^4}{R_2^4} - 0,092 \frac{r^6}{R_2^6} - 0,00189 \frac{R_2^8}{r^8} + 0,0147 \frac{R_2^6}{r^6} \right) \sin 6\Theta + \\
 & + \left(-0,01302 \frac{r^5}{R_2^5} + 0,01484 \frac{r^7}{R_2^7} + 0,00027 \frac{R_2^9}{r^9} - 0,0021 \frac{R_2^7}{r^7} \right) \sin 7\Theta + \\
 & + \left(0,0023 \frac{r^6}{R_2^6} - 0,00259 \frac{r^8}{R_2^8} - 0,000040 \frac{R_2^{10}}{r^{10}} + 0,000325 \frac{R_2^8}{r^8} \right) \sin 8\Theta + \\
 & \left. + \left(-0,00035 \frac{r^7}{R_2^7} + 0,000388 \frac{r^9}{R_2^9} + 0,0000055 \frac{R_2^{11}}{r^{11}} - 0,0000439 \frac{R_2^9}{r^9} \right) \sin 9\Theta \right] \quad (47)
 \end{aligned}$$

NOTICE:

1) In formulas (45) and (47) the sum of the numerical factors is equal to zero for any value of $n > 1$ (in sufficient approximation depending on the number of decimals given for $a_n, b_n, \alpha_n, \beta_n$). This condition corresponds to zero values of $\widehat{rr}$ and $\widehat{r\Theta}$ stresses on the external boundary of the ring, when $r = R_2$.

2) As follows from formulas (45, 46, 47), terms with λ and γ values depend on the ratio $\frac{\lambda}{\gamma_l}$; thus only μ — Poisson's ratio, remains in these formulas, and modulus of elasticity — E will be cancelled.

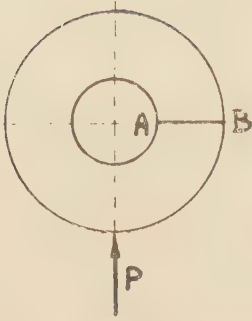


Fig. 9.

To check the results obtained the cross section $A - B$ will be considered. It corresponds to $\angle \Theta = \frac{\pi}{2}$. As follows from the conditions of symmetry, the value of a normal force acting on $A - B$ is equal to $-\frac{P}{4}$.

Value of $\widehat{\Theta\Theta}$ in this cross section may be found from (46) after the substitution $\Theta = \frac{\pi}{2}$

$$\begin{aligned} (\widehat{\Theta\Theta})_{\Theta=\frac{\pi}{2}} = & -\frac{P}{16\pi r} \left(\frac{R_2}{r} + 9 \frac{r}{R_2} \right) + \frac{P}{\pi R_2} \cdot \frac{\frac{r^4}{R_2^4} + 2 \frac{r^2}{R_2^2} + 1}{\frac{r^4}{R_2^4} + 2 \frac{r^2}{R_2^2} + 1} + \\ & + \frac{P}{54\pi R_2} \left(-33,36 + 99,132 \frac{r^2}{R_2^2} - 3,769 \frac{r^4}{R_2^4} - 3,228 \frac{R_2^4}{r^4} + 0,1249 \frac{r^6}{R_2^6} + \right. \\ & \left. + 0,0858 \frac{R_2^6}{r^6} - 0,00324 \frac{r^8}{R_2^8} - 0,00213 \frac{R_2^8}{r^8} + 0,000040 \frac{R_2^{10}}{r^{10}} \right) \end{aligned} \quad (48)$$

$$\begin{aligned} \text{thus } \int_{\frac{R_2}{3}}^{R_2} (\widehat{\Theta\Theta})_{\Theta=\frac{\pi}{2}} dr = & -\frac{P}{8\pi} - \frac{3P}{8\pi} + \frac{P}{\pi} (0,4667 - 0,1476\pi) + \frac{P}{54\pi} (-22,24 + 31,83 + \\ & - 0,754 - 27,976 + 0,0178 + 4,152 - 0,0004 - 0,665 + 0,0875) = -0,2499 \cdot P \cong -\frac{P}{4} \end{aligned}$$

This checks the correctness of the previous formulas. The values of principal stresses may be easily found when $\widehat{rr}$, $\widehat{\Theta\Theta}$, $\widehat{r\Theta}$, stresses are known.

The method applied in the previous case may be used also when the concentrated force P is applied on the internal boundary of the ring.

As follows from the previous considerations, the influence of body forces on the magnitude of internal stresses is expressed by terms of single angle Θ . When force P is replaced by external loading as given by formulas (27, 27a), the values of coefficients a_0, b_0, c_0 and $a_1, b_1, \alpha_1, \beta_1$,

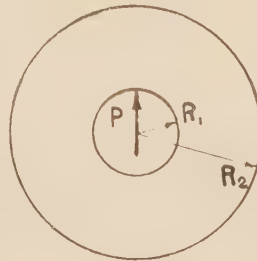


Fig. 10.

may be easily determined. With the increasing radius $R_2 \rightarrow \infty$, values of $m = \frac{R_2}{R_1} \rightarrow \infty$, and all other coefficients may be found from (33). Thus the methods of calculation are similar for both cases, and external loading for the ring with $R_2 \rightarrow \infty$ may be taken in the same way as for the ring with $R_2 = \text{constant}$ (to each value of $n > 1$ there corresponds an external loading in equilibrium).

IV. Stresses in a circular ring, compressed by two opposite forces acting along a diameter.

The solution of the problem with a single concentrated force P may be used also for the ring under the action of two concentrated forces P acting in a diametrical direction, when the body forces of the ring are neglected. Two rings as shown in Fig. 11 are superimposed on each other. Then the values of stresses in formulas (45, 46, 47) must be added to the same values, where $\angle(\theta - \pi)$ is substituted in place of the $\angle \theta$, and the results obtained answer this problem. The following formulas are obtained in this way:

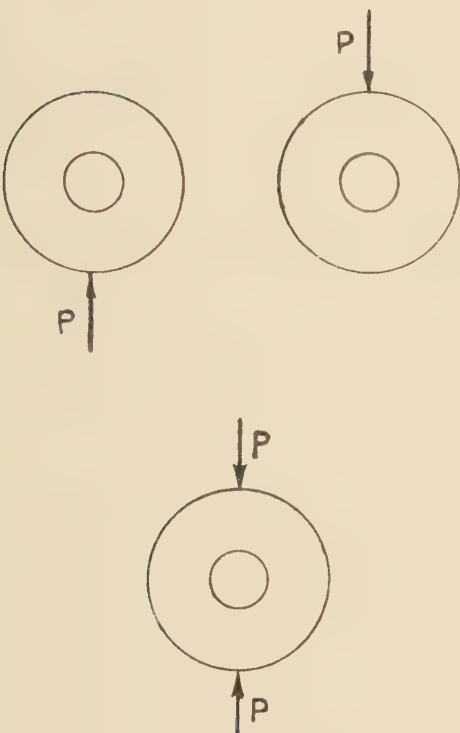


Fig. 11.

$$\begin{aligned}
 \sigma_r = & \frac{P}{8 \pi r} \left(\frac{R_2}{r} - 9 \frac{r}{R_2} \right) + \\
 & + \frac{P}{2 \pi R_2} \left\{ \frac{1 - \frac{r^2}{R_2^2}}{\frac{r}{R_2}} \left[\frac{\left(\frac{r^2}{R_2^2} + 1 \right) \cos \theta + 2 \frac{r}{R_2}}{\left(1 + 2 \frac{r}{R_2} \cos \theta + \frac{r^2}{R_2^2} \right)^2} + \right. \right. \\
 & + \left. \frac{2 \frac{r}{R_2} - \left(\frac{r^2}{R_2^2} + 1 \right) \cos \theta}{\left(1 - 2 \frac{r}{R_2} \cos \theta + \frac{r^2}{R_2^2} \right)^2} \right] + \\
 & + 2 \frac{r}{R_2} \left[\frac{\cos \theta + \frac{r}{R_2}}{1 + 2 \frac{r}{R_2} \cos \theta + \frac{r^2}{R_2^2}} + \right. \\
 & + \left. \frac{\frac{r}{R_2} - \cos \theta}{1 - 2 \frac{r}{R_2} \cos \theta + \frac{r^2}{R_2^2}} \right] \Bigg\} + \frac{P}{27 \pi R_2} \left[\left(-33,36 - 2,952 \frac{R_2^4}{r^4} + 36,32 \frac{R_2^2}{r^2} \right) \cos 2\theta + \right. \\
 & + \left(-1,992 \frac{r^2}{R_2^2} + 1,23 \frac{r^4}{R_2^4} - 0,076 \frac{R_2^6}{r^6} + 0,830 \frac{R_2^4}{r^4} \right) \cos 4\theta + \left(-0,079 \frac{r^4}{R_2^4} + \right. \\
 & + \left. 0,0613 \frac{r^6}{R_2^6} - 0,00189 \frac{R_2^8}{r^8} + 0,0196 \frac{R_2^6}{r^6} \right) \cos 6\theta + \left(-0,0023 \frac{r^6}{R_2^6} + 0,00194 \frac{r^8}{R_2^8} - \right. \\
 & \left. \left. - 0,000040 \frac{R_2^{10}}{r^{10}} + 0,000406 \frac{R_2^8}{r^8} \right) \cos 8\theta \right]
 \end{aligned} \quad (49)$$

$$\begin{aligned}
 \Theta\Theta = & -\frac{P}{8\pi r} \left(\frac{R_2}{r} + 9 \frac{r}{R_2} \right) - \frac{P}{2\pi R_2} \left\{ \frac{1 - \frac{r^2}{R_2^2}}{\frac{r}{R_2}} \left[\frac{\left(\frac{r^2}{R_2^2} + 1 \right) \cos \Theta + 2 \frac{r}{R_2}}{\left(1 + 2 \frac{r}{R_2} \cos \Theta + \frac{r^2}{R_2^2} \right)^2} + \right. \right. \\
 & + \left. \frac{2 \frac{r}{R_2} - \left(\frac{r^2}{R_2^2} + 1 \right) \cos \Theta}{\left(1 - 2 \frac{r}{R_2} \cos \Theta + \frac{r^2}{R_2^2} \right)^2} \right] - 2 \frac{r}{R_2} \left[\frac{\cos \Theta + \frac{r}{R_2}}{1 + 2 \frac{r}{R_2} \cos \Theta + \frac{r^2}{R_2^2}} + \right. \\
 & + \left. \left. \frac{\frac{r}{R_2} - \cos \Theta}{1 - 2 \frac{r}{R_2} \cos \Theta + \frac{r^2}{R_2^2}} \right] \right\} + \frac{P}{27\pi R_2} \left[\left(33,36 + 2,952 \frac{R_2^4}{r^4} - 97,14 \frac{r^2}{R_2^2} \right) \cos 2\Theta + \right. \\
 & + \left(1,992 \frac{r^2}{R_2^2} - 3,69 \frac{r^4}{R_2^4} + 0,076 \frac{R_2^6}{r^6} - 0,276 \frac{R_2^4}{r^4} \right) \cos 4\Theta + \\
 & + \left(0,079 \frac{r^4}{R_2^4} - 0,1226 \frac{r^6}{R_2^6} + 0,00189 \frac{R_2^8}{r^8} - 0,0098 \frac{R_2^6}{r^6} \right) \cos 6\Theta + \\
 & + \left. \left(0,0023 \frac{r^6}{R_2^6} - 0,00324 \frac{r^8}{R_2^8} + 0,000040 \frac{R_2^{10}}{r^{10}} - 0,00024 \frac{R_2^8}{r^8} \right) \cos 8\Theta \right] \quad (50)
 \end{aligned}$$

$$\begin{aligned}
 \widehat{r\Theta} = & \frac{P}{2\pi R_2} \cdot \frac{1 - \frac{r^2}{R_2^2}}{\frac{r}{R_2}} \left[\frac{\frac{r^2}{R_2^2} - 1}{\left(1 + 2 \frac{r}{R_2} \cos \Theta + \frac{r^2}{R_2^2} \right)^2} - \frac{\frac{r^2}{R_2^2} - 1}{\left(1 - 2 \frac{r}{R_2} \cos \Theta + \frac{r^2}{R_2^2} \right)^2} \right] \sin \Theta + \\
 & + \frac{P}{27\pi R_2} \left[\left(33,36 - 48,57 \frac{r^2}{R_2^2} - 2,952 \frac{R_2^4}{r^4} + 18,16 \frac{R_2^2}{r^2} \right) \sin 2\Theta + \right. \\
 & + \left(1,992 \frac{r^2}{R_2^2} - 2,46 \frac{r^4}{R_2^4} - 0,076 \frac{R_2^6}{r^6} + 0,552 \frac{R_2^4}{r^4} \right) \sin 4\Theta + \\
 & + \left(0,079 \frac{r^4}{R_2^4} - 0,092 \frac{r^6}{R_2^6} - 0,00189 \frac{R_2^8}{r^8} + 0,0147 \frac{R_2^6}{r^6} \right) \sin 6\Theta + \\
 & + \left. \left(0,0023 \frac{r^6}{R_2^6} - 0,00259 \frac{r^8}{R_2^8} - 0,000040 \frac{R_2^{10}}{r^{10}} + 0,000325 \frac{R_2^8}{r^8} \right) \sin 8\Theta \right] \quad (51)
 \end{aligned}$$

The previous solution, obtained by the general method for the ring under the action of two concentrated forces, may be obtained also from Flamant's solution. For a plate as shown in (fig. 12) $\sigma_r = \frac{-2P}{\pi b} \cdot \frac{\cos \alpha}{r}$. For a disk with two concentrated forces (Fig. 13) the solution may be obtained as a sum of stresses in a plate (fig. 12) and in another similar plate inverted 180° superimposed on it. The stress function Φ for this case may be written as follows¹⁾:

$$\Phi = \frac{P}{\pi} \left(\frac{r^2}{2R_2} - r_1 \alpha_1 \sin \alpha_1 - r_2 \alpha_2 \sin \alpha_2 \right) \quad (52)$$

The stresses induced by P consist of three parts:

1) The uniform tensile stress on the boundary of the disk:

$$\sigma_1 = \frac{P}{\pi R_2} \quad (53a)$$

2) The compression in the direction of r_1 :

$$\sigma_2 = - \frac{2P \cos \alpha_1}{\pi r_1} \quad (53b)$$

3) The compression in the direction of r_2 :

$$\sigma_3 = - \frac{2P \cos \alpha_2}{\pi r_2} \quad (53c)$$

A unit width of the disk is assumed. To find the influence of a concentric hole in the disk on the value of stresses, it's necessary to express the stresses on the boundary of this hole in terms of the radius.

The normal stress at point A (Fig. 14) is:

$$\sigma_n = \sigma_1 + \sigma_2 \sin^2 \beta_1 + \sigma_3 \sin^2 \beta_2$$

$$\text{or } \sigma_n = \frac{2P}{\pi} \left(\frac{1}{2R_2} - \frac{\cos \alpha_1 \cdot \sin^2 \beta_1}{r_1} - \frac{\cos \alpha_2 \cdot \sin^2 \beta_2}{r_2} \right) \quad (54)$$

The shearing stress at point A is:

$$\tau = - (\sigma_2 \sin \beta_1 \cos \beta_1 + \sigma_3 \sin \beta_2 \cos \beta_2) = \left. \begin{aligned} &= \frac{P}{\pi} \left[\frac{\cos \alpha_1 \sin 2\beta_1}{r_1} + \frac{\cos \alpha_2 \sin 2\beta_2}{r_2} \right] \end{aligned} \right\} \quad (55)$$

where β_1 is measured from r_1 in a clockwise direction; similarly with β_2 .

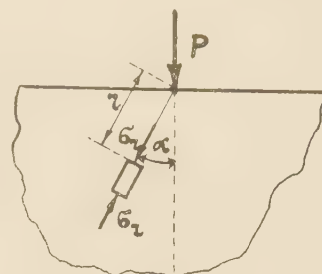


Fig. 12.

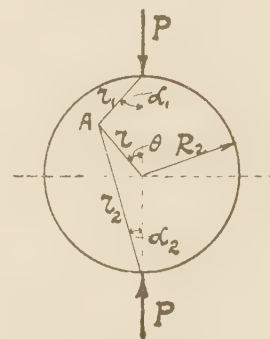


Fig. 13.

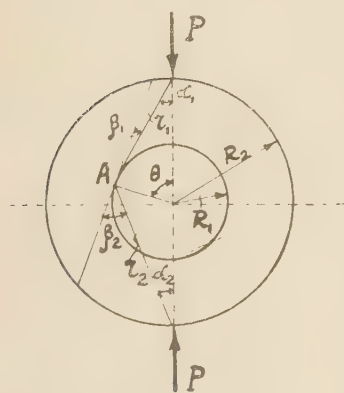


Fig. 14.

¹⁾ S. P. Timoshenko. Bulletin of Kiev Polytechnic Institute. "Compression of a ring by two equal opposite forces".

From geometric considerations the following relations are obtained:

$$\begin{aligned}
 r_1^2 &= (R_2 - R_1 \cos \Theta)^2 + R_1^2 \sin^2 \Theta = R_2^2 + R_1^2 - 2 R_2 R_1 \cos \Theta \\
 r_2^2 &= (R_2 + R_1 \cos \Theta)^2 + R_1^2 \sin^2 \Theta = R_2^2 + R_1^2 + 2 R_2 R_1 \cos \Theta \\
 \cos \alpha_1 &= \frac{R_2 - R_1 \cos \Theta}{r_1}, & \cos \alpha_2 &= \frac{R_2 + R_1 \cos \Theta}{r_2} \\
 \sin \beta_1 &= -\frac{R_1 - R_2 \cos \Theta}{r_1}, & \sin \beta_2 &= \frac{R_1 + R_2 \cos \Theta}{r_2} \\
 \cos \beta_1 &= \frac{R_2 \sin \Theta}{r_1}, & \cos \beta_2 &= \frac{R_2 \sin \Theta}{r_2}
 \end{aligned}$$

After the substitution of these values into the formulas (54) and (55) there will be:

$$\begin{aligned}
 \sigma_n &= \frac{2P}{\pi R_2} \left[\frac{1}{2} - \frac{\left(1 - \frac{R_1}{R_2} \cos \Theta\right) \left(\frac{R_1}{R_2} - \cos \Theta\right)^2}{\left[1 + \left(\frac{R_1}{R_2}\right)^2 - 2 \frac{R_1}{R_2} \cos \Theta\right]^2} - \frac{\left(1 + \frac{R_1}{R_2} \cos \Theta\right) \left(\frac{R_1}{R_2} + \cos \Theta\right)^2}{\left[1 + \left(\frac{R_1}{R_2}\right)^2 + 2 \frac{R_1}{R_2} \cos \Theta\right]^2} \right] \\
 \tau &= \frac{2P}{\pi R_2} \sin \Theta \left[\frac{\left(1 - \frac{R_1}{R_2} \cos \Theta\right) \left(\cos \Theta - \frac{R_1}{R_2}\right)}{\left[1 + \left(\frac{R_1}{R_2}\right)^2 - 2 \frac{R_1}{R_2} \cos \Theta\right]^2} + \frac{\left(1 + \frac{R_1}{R_2} \cos \Theta\right) \left(\cos \Theta + \frac{R_1}{R_2}\right)}{\left[1 + \left(\frac{R_1}{R_2}\right)^2 + 2 \frac{R_1}{R_2} \cos \Theta\right]^2} \right]
 \end{aligned} \quad (56)$$

The same formulas may be used for any circumference with r as a radius; then in formulas (56) R_1 should be changed into r . To find the stresses in a circular ring under the action of two opposite forces, acting along a diameter, the stresses $-\sigma_n$ and $-\tau$ should be applied on the boundary of the inner circle. Then the total stress is equal to that in a solid disk plus the influence of $-\sigma_n$ and $-\tau$ stresses on the inner boundary of the ring.

For the present case it is assumed $\frac{R_2}{R_1} = 3$. Formulas (56) for σ_n and τ may be written:

$$\left. \begin{aligned} \sigma_n &= -\frac{2P}{\pi R_2} k \\ \tau &= +\frac{2P}{\pi R_2} l \end{aligned} \right\} \dots \dots \dots (56a)$$

where factors k and l are functions of the angle Θ . The following tables contain values of these factors.

$$k = -\frac{1}{2} + \frac{3(3 - \cos \Theta)(1 - 3 \cos \Theta)^2}{(10 - 6 \cos \Theta)^2} + \frac{3(3 + \cos \Theta)(1 + 3 \cos \Theta)^2}{(10 + 6 \cos \Theta)^2}$$

Values of factor k for different angles.

Θ	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
k	1.750	1.6183	1.2799	0.8587	0.4636	0.1496	-0.0734	-0.2166	-0.2951	-0.320

Due to the fact that values of σ_n are symmetrical with respect to the vertical and horizontal diameters, the same value of normal stress exists for the angles Θ , $\pi - \Theta$, $\pi + \Theta$, $2\pi - \Theta$. Therefore the development into Fourier's series may be taken in terms of cosines of even angles only.

$$k = m_0 + m_2 \cdot \cos 2\Theta + m_4 \cdot \cos 4\Theta + m_6 \cdot \cos 6\Theta + m_8 \cdot \cos 8\Theta + m_{10} \cdot \cos 10\Theta \quad (57)$$

To find values of factors m_n the whole distance from 0° to 360° is divided into 36 equal parts; then:

$$m_0 = \frac{1}{18} \sum_{18} k_{\Theta}, \quad m_n = \frac{1}{9} \sum_{18} k_{\Theta} \cdot \cos (n\Theta)$$

where $\Theta = 0^\circ, 10^\circ, 20^\circ \dots 170^\circ$

from which the values of m are obtained

m_0	m_2	m_4	m_6	m_8	m_{10}
0,5000	1,0000	0,2099	0,0343	0,00504	0,000684

and the expression for k may be written:

$$k = 0,5000 + 1,000 \cdot \cos 2\Theta + 0,210 \cdot \cos 4\Theta + 0,0343 \cdot \cos 6\Theta + \left. \begin{aligned} &+ 0,0050 \cdot \cos 8\Theta + 0,00068 \cdot \cos 10\Theta \end{aligned} \right\} \quad (57a)$$

Values of factor l .

$$l = 9 \sin \Theta \left[\frac{(3 - \cos \Theta)(3 \cos \Theta - 1)}{(10 - 6 \cos \Theta)^2} + \frac{(3 + \cos \Theta)(3 \cos \Theta + 1)}{(10 + 6 \cos \Theta)^2} \right]$$

Θ	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
l	0,0000	0,4651	0,7958	0,9361	0,9118	0,7828	0,6024	0,4035	0,2014	0,0000

Development of l into Fourier's series may be taken in terms of sines of even angles only, for the shearing stress τ has the same value for angles Θ and $\pi + \Theta$, but which is of the opposite sign to τ for $\pi - \Theta$ and $2\pi - \Theta$.

$$l = p_2 \sin 2\Theta + p_4 \sin 4\Theta + p_6 \sin 6\Theta + p_8 \sin 8\Theta + p_{10} \sin 10\Theta \dots (58)$$

The factors p_n are found from the formula:

$$p_n = \frac{1}{9} \sum_{18} l_{\Theta} \cdot \sin (n\Theta)$$

where $\Theta = 0^\circ, 10^\circ, 20^\circ \dots 170^\circ$

P_2	P_4	P_6	P_8	P_{10}
0,8889	0,1975	0,03295	0,00488	0,000684

and the expression for l may be written:

$$l = 0,8889 \cdot \sin 2\Theta + 0,1975 \cdot \sin 4\Theta + 0,03295 \cdot \sin 6\Theta + \left. \begin{aligned} &+ 0,00488 \cdot \sin 8\Theta + 0,000684 \cdot \sin 10\Theta \end{aligned} \right\} \dots \quad (58a)$$

Now when stresses $-\sigma_n$ and $-\tau$ are expressed in terms of Fourier's series it is easy to find the corresponding internal stresses (Formulas 13a-b-c). Values of factors a_n , b_n , α_n , β_n are given by formulas (44) where for the present case

$$A_n = \frac{2P}{\pi R_2} m_n$$

$$D_n = -\frac{2P}{\pi R_2} p_n$$

Values of factors are given in the next tables. The value of $m_0 = 0,5$ corresponds to the uniform compression on the inner boundary of the ring, and corresponding internal stresses are given by Lamé's formula:

$$\left. \begin{aligned} \widehat{rr} &= 0,5 \frac{2P}{\pi R_2} \cdot \frac{R_1^2}{r^2} \cdot \frac{R_2^2 - r^2}{R_2^2 - R_1^2} \\ \widehat{\Theta\Theta} &= -0,5 \frac{2P}{\pi R_2} \cdot \frac{R_1^2}{r^2} \cdot \frac{R_2^2 + r^2}{R_2^2 - R_1^2} \end{aligned} \right\} \dots \quad (59)$$

Values of factors a_n , b_n , α_n , β_n

$n =$	2	4	6	8	10
$a_n =$	$6172 \cdot 10^{-4} \frac{P}{\pi R_2}$	$6120 \cdot 10^{-6} \frac{P}{\pi R_2^3}$	$9875 \cdot 10^{-8} \frac{P}{\pi R_2^5}$	$1540 \cdot 10^{-9} \frac{P}{\pi R_2^7}$	$2290 \cdot 10^{-11} \frac{P}{\pi R_2^9}$
$b_n =$	$-2995 \cdot 10^{-4} \frac{P}{\pi R_2^3}$	$-4555 \cdot 10^{-6} \frac{P}{\pi R_2^5}$	$-8201 \cdot 10^{-8} \frac{P}{\pi R_2^7}$	$-1344 \cdot 10^{-9} \frac{P}{\pi R_2^9}$	$-2059 \cdot 10^{-11} \frac{P}{\pi R_2^{11}}$
$\alpha_n =$	$1823 \cdot 10^{-5} \frac{PR_2^3}{\pi}$	$1414 \cdot 10^{-7} \frac{PR_2^5}{\pi}$	$1703 \cdot 10^{-9} \frac{PR_2^7}{\pi}$	$2096 \cdot 10^{-11} \frac{PR_2^9}{\pi}$	$2574 \cdot 10^{-13} \frac{PR_2^{11}}{\pi}$
$\beta_n =$	$-3359 \cdot 10^{-4} \frac{PR_2}{\pi}$	$1707 \cdot 10^{-6} \frac{PR_2^3}{\pi}$	$1845 \cdot 10^{-8} \frac{PR_2^5}{\pi}$	$2160 \cdot 10^{-10} \frac{PR_2^7}{\pi}$	$2574 \cdot 10^{-12} \frac{PR_2^9}{\pi}$

Values of internal stresses $\widehat{rr}$, $\widehat{\Theta\Theta}$, $\widehat{r\Theta}$, in the ring will be obtained as a sum of internal stresses in the disk, and internal stresses in the ring under the action of $-\sigma_n$ and $-\tau$ stresses on the inner boundary of the ring. Values of $\widehat{rr}$ and $\widehat{r\Theta}$ stresses for the disk are given by formulas (56) (changing R_1 into r); in a similar manner values of $\widehat{\Theta\Theta}$ stresses are found.

$$(\Theta\Theta)_{\text{disk}} = \frac{P}{\pi R_2} - \frac{2P \cos \alpha_1}{\pi r_1} \cos^2 \beta_1 - \frac{2P \cos \alpha_2}{\pi r_2} \cos^2 \beta_2 \quad . \quad . \quad . \quad (60)$$

or after the substitution of values:

$$(\Theta\Theta)_{\text{disk}} = \frac{2P}{\pi R_2} \left[\frac{1}{2} - \frac{\left(1 - \frac{r}{R_2} \cos \Theta\right) \sin^2 \Theta}{\left[1 + \left(\frac{r}{R_2}\right)^2 - 2 \frac{r}{R_2} \cos \Theta\right]^2} - \frac{\left(1 + \frac{r}{R_2} \cos \Theta\right) \sin^2 \Theta}{\left[1 + \left(\frac{r}{R_2}\right)^2 + 2 \frac{r}{R_2} \cos \Theta\right]^2} \right] \quad . \quad . \quad (60a)$$

The final formulas of stresses are obtained as a summation of results:

$$\begin{aligned} \widehat{rr} = \frac{2P}{\pi R_2} & \left[\frac{1}{2} - \frac{\left(1 - \frac{r}{R_2} \cos \Theta\right) \left(\frac{r}{R_2} - \cos \Theta\right)^2}{\left[1 + \left(\frac{r}{R_2}\right)^2 - 2 \frac{r}{R_2} \cos \Theta\right]^2} + \right. \\ & - \frac{\left(1 + \frac{r}{R_2} \cos \Theta\right) \left(\frac{r}{R_2} + \cos \Theta\right)^2}{\left[1 + \left(\frac{r}{R_2}\right)^2 + 2 \frac{r}{R_2} \cos \Theta\right]^2} + \frac{1}{16} \cdot \frac{R_2^2 - r^2}{r^2} + \\ & + \left(-0,6172 - 0,05469 \frac{R_2^4}{r^4} + 0,67188 \frac{R_2^2}{r^2} \right) \cos 2\Theta + \\ & + \left(-0,03672 \frac{r^2}{R_2^2} + 0,022775 \frac{r^4}{R_2^4} - 0,001414 \frac{R_2^6}{r^6} + 0,015363 \frac{R_2^4}{r^4} \right) \cos 4\Theta + \\ & + \left(-0,001481 \frac{r^4}{R_2^4} + 0,001148 \frac{r^6}{R_2^6} + \right. \\ & - 0,00003576 \frac{R_2^8}{r^8} + 0,0003690 \frac{R_2^6}{r^6} \left. \right) \cos 6\Theta + \\ & + \left(-0,00004312 \frac{r^6}{R_2^6} + 0,00003629 \frac{r^8}{R_2^8} + \right. \\ & - 0,0000007546 \frac{R_2^{10}}{r^{10}} + 0,000007560 \frac{R_2^8}{r^8} \left. \right) \cos 8\Theta + \\ & + \left(-0,000001031 \frac{r^8}{R_2^8} + 0,0000009060 \frac{r^{10}}{R_2^{10}} + \right. \\ & - 0,00000001416 \frac{R_2^{12}}{r^{12}} + 0,0000001390 \frac{R_2^{10}}{r^{10}} \left. \right) \cos 10\Theta \left. \right] \quad (61) \end{aligned}$$

$$\begin{aligned}
 \widehat{\Theta\Theta} = \frac{2P}{\pi R_2} & \left[\frac{1}{2} - \frac{\left(1 - \frac{r}{R_2} \cos \Theta\right) \sin^2 \Theta}{\left[1 + \left(\frac{r}{R_2}\right)^2 - 2 \frac{r}{R_2} \cos \Theta\right]^2} - \frac{\left(1 + \frac{r}{R_2} \cos \Theta\right) \sin^2 \Theta}{\left[1 + \left(\frac{r}{R_2}\right)^2 + 2 \frac{r}{R_2} \cos \Theta\right]^2} + \right. \\
 & - \frac{1}{16} \cdot \frac{R_2^2 + r^2}{r^2} + \left(0,6172 - 1,7969 \frac{r^2}{R_2^2} + 0,05469 \frac{R_2^4}{r^4}\right) \cos 2\Theta + \\
 & + \left(0,03672 \frac{r^2}{R_2^2} - 0,06833 \frac{r^4}{R_2^4} + 0,001414 \frac{R_2^6}{r^6} - 0,005121 \frac{R_2^4}{r^4}\right) \cos 4\Theta + \\
 & + \left(0,001481 \frac{r^4}{R_2^4} - 0,002296 \frac{r^6}{R_2^6} + 0,00003576 \frac{R_2^8}{r^8} - 0,0001845 \frac{R_2^6}{r^6}\right) \cos 6\Theta + \\
 & + \left(0,00004312 \frac{r^6}{R_2^6} - 0,00006048 \frac{r^8}{R_2^8} + 0,0000007546 \frac{R_2^{10}}{r^{10}} + \right. \\
 & \left. - 0,000004536 \frac{R_2^8}{r^8}\right) \cos 8\Theta + \left(0,000001031 \frac{r^8}{R_2^8} - 0,000001359 \frac{r^{10}}{R_2^{10}} + \right. \\
 & \left. + 0,00000001416 \frac{R_2^{12}}{r^{12}} - 0,00000009266 \frac{R_2^{10}}{r^{10}}\right) \cos 10\Theta \Big]
 \end{aligned} \tag{62}$$

$$\begin{aligned}
 \widehat{r\Theta} = \frac{2P}{\pi R_2} & \left\{ \left[\frac{\left(1 - \frac{r}{R_2} \cos \Theta\right) \left(\cos \Theta - \frac{r}{R_2}\right)}{\left[1 + \left(\frac{r}{R_2}\right)^2 - 2 \frac{r}{R_2} \cos \Theta\right]^2} + \frac{\left(1 + \frac{r}{R_2} \cos \Theta\right) \left(\cos \Theta + \frac{r}{R_2}\right)}{\left[1 + \left(\frac{r}{R_2}\right)^2 + 2 \frac{r}{R_2} \cos \Theta\right]^2} \right] \sin \Theta + \right. \\
 & + \left(0,6172 - 0,8984 \frac{r^2}{R_2^2} - 0,05469 \frac{R_2^4}{r^4} + 0,33594 \frac{R_2^2}{r^2}\right) \sin 2\Theta + \\
 & + \left(0,03672 \frac{r^2}{R_2^2} - 0,04555 \frac{r^4}{R_2^4} - 0,001414 \frac{R_2^6}{r^6} + 0,01024 \frac{R_2^4}{r^4}\right) \sin 4\Theta + \\
 & + \left(0,001481 \frac{r^4}{R_2^4} - 0,001722 \frac{r^6}{R_2^6} - 0,00003576 \frac{R_2^8}{r^8} + 0,0002768 \frac{R_2^6}{r^6}\right) \sin 6\Theta + \\
 & + \left(0,00004312 \frac{r^6}{R_2^6} - 0,00004838 \frac{r^8}{R_2^8} + \right. \\
 & \left. - 0,0000007546 \frac{R_2^{10}}{r^{10}} + 0,000006048 \frac{R_2^8}{r^8}\right) \sin 8\Theta + \left(0,000001031 \frac{r^8}{R_2^8} + \right. \\
 & \left. - 0,000001132 \frac{r^{10}}{R_2^{10}} - 0,00000001416 \frac{R_2^{12}}{r^{12}} + 0,0000001158 \frac{R_2^{10}}{r^{10}}\right) \sin 10\Theta \Big\}
 \end{aligned} \tag{63}$$

To check the correctness of these formulas the cross section $A - B$ will be considered (Fig. 15). Due to conditions of symmetry the value of a normal force should be equal to $-\frac{P}{2}$

thus

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\bar{\Theta}\bar{\Theta})_{\Theta=\frac{\pi}{2}} dr = -\frac{P}{2}$$



Fig. 15.

The value of $\bar{\Theta}\bar{\Theta}$ stress in cross section $A - B$ may be found from (62) after the substitution: $\Theta = \frac{\pi}{2}$

$$\begin{aligned} (\bar{\Theta}\bar{\Theta})_{\Theta=\frac{\pi}{2}} = \frac{2P}{\pi R_2} & \left[0,5 - \frac{2 R_2^4}{(R_2^2 + r^2)^2} - \frac{1}{16} \cdot \frac{R_2^2 + r^2}{r^2} - 0,6172 + 1,8336 \frac{r^2}{R_2^2} + \right. \\ & - 0,06981 \frac{r^4}{R_2^4} + 0,002339 \frac{r^6}{R_2^6} - 0,00006151 \frac{r^8}{R_2^8} + 0,000001359 \frac{r^{10}}{R_2^{10}} - 0,05981 \frac{R_2^4}{r^4} + \\ & \left. + 0,001599 \frac{R_2^6}{r^6} - 0,00004030 \frac{R_2^8}{r^8} + 0,0000008473 \frac{R_2^{10}}{r^{10}} - 0,00000001416 \frac{R_2^{12}}{r^{12}} \right] \end{aligned}$$

$$\begin{aligned} \int_{\frac{R_1}{3}}^{R_2} (\bar{\Theta}\bar{\Theta})_{\Theta=\frac{\pi}{2}} dr = \frac{2P}{\pi R_2} & \left[0,5 \cdot r \cdot \frac{R_2}{3} - 0,6172 \cdot r \cdot \frac{R_2}{3} - R_2 \cdot \arctg \frac{r}{R_2} \cdot \frac{R_2}{3} - \frac{R_2^2 \cdot r}{R_2^2 + r^2} \cdot \frac{R_2}{3} + \right. \\ & + 0,0625 \cdot \frac{R_2^2}{r} \cdot \frac{R_2}{3} - 0,0625 \cdot r \cdot \frac{R_2}{3} + 1,8336 \cdot \frac{1}{3} \cdot \frac{r^3}{R_2^2} \cdot \frac{R_2}{3} - 0,06981 \cdot \frac{1}{5} \cdot \frac{r^5}{R_2^4} \cdot \frac{R_2}{3} + \\ & + 0,002339 \cdot \frac{1}{7} \cdot \frac{r^7}{R_2^6} \cdot \frac{R_2}{3} - 0,00006151 \cdot \frac{1}{9} \cdot \frac{r^9}{R_2^8} \cdot \frac{R_2}{3} + 0,000001359 \cdot \frac{1}{11} \cdot \frac{r^{11}}{R_2^{10}} \cdot \frac{R_2}{3} + \\ & + 0,05981 \cdot \frac{1}{3} \cdot \frac{R_2^4}{r^3} \cdot \frac{R_2}{3} - 0,001599 \cdot \frac{1}{5} \cdot \frac{R_2^6}{r^5} \cdot \frac{R_2}{3} + 0,00004030 \cdot \frac{1}{7} \cdot \frac{R_2^8}{r^7} \cdot \frac{R_2}{3} + \\ & \left. - 0,0000008473 \cdot \frac{1}{9} \cdot \frac{R_2^{10}}{r^9} \cdot \frac{R_2}{3} + 0,00000001416 \cdot \frac{1}{11} \cdot \frac{R_2^{12}}{r^{11}} \cdot \frac{R_2}{3} \right] \end{aligned}$$

or

$$\int_{\frac{R_1}{3}}^{R_2} (\bar{\Theta}\bar{\Theta})_{\Theta=\frac{\pi}{2}} dr = -\frac{2P \cdot 0,785383}{3,141593} = -0,49999 \cdot P$$

which agrees with the conditions of the external loading.

NOTICE:

The formulas (49, 50, 51) and (61, 62, 63) are identical (with sufficient approximation) as solutions of the same problem. This may be easily checked after some algebraical manipulations have been done.

The value of $\Theta\Theta$ stress in a vertical radial cross section is:

$$(\Theta\Theta)_{\Theta=0} = \frac{2P}{\pi R_2} \left[1,1172 - \frac{1}{16} \cdot \frac{R_2^2 + r^2}{r^2} - 1,7602 \frac{r^2}{R_2^2} + 0,04957 \frac{R_2^4}{r^4} - 0,06685 \frac{r^4}{R_2^4} + \right. \\ \left. + 0,001230 \frac{R_2^6}{r^6} - 0,002253 \frac{r^6}{R_2^6} + 0,00003122 \frac{R_2^8}{r^8} - 0,00005945 \frac{r^8}{R_2^8} + 0,0000006619 \frac{R_2^{10}}{r^{10}} + \right. \\ \left. - 0,000001359 \frac{r^{10}}{R_2^{10}} + 0,00000001416 \frac{R_2^{12}}{r^{12}} \right]$$

As follows from the conditions of equilibrium, the value of the normal force in a vertical radial cross section should be equal to zero. Now the value of $\int_{\frac{R_2}{3}}^{R_2} (\Theta\Theta)_{\Theta=0} dr$ will be considered.

$$\int_{\frac{R_2}{3}}^{R_2} (\Theta\Theta)_{\Theta=0} dr = \frac{2P}{\pi R_2} \left[1,1172 \cdot r \Big|_{\frac{R_2}{3}}^{R_2} + 0,0625 \frac{R_2^2}{r} \Big|_{\frac{R_2}{3}}^{R_2} - 0,0625 \cdot r \Big|_{\frac{R_2}{3}}^{R_2} - 1,7602 \cdot \frac{1}{3} \cdot \frac{r^3}{R_2^2} \Big|_{\frac{R_2}{3}}^{R_2} + \right. \\ \left. - 0,06685 \cdot \frac{1}{5} \cdot \frac{r^5}{R_2^4} \Big|_{\frac{R_2}{3}}^{R_2} - 0,002253 \cdot \frac{1}{7} \cdot \frac{r^7}{R_2^6} \Big|_{\frac{R_2}{3}}^{R_2} - 0,00005945 \cdot \frac{1}{9} \cdot \frac{r^9}{R_2^8} \Big|_{\frac{R_2}{3}}^{R_2} + \right. \\ \left. - 0,000001359 \cdot \frac{1}{11} \cdot \frac{r^{11}}{R_2^{10}} \Big|_{\frac{R_2}{3}}^{R_2} - 0,04957 \cdot \frac{1}{3} \cdot \frac{R_2^4}{r^3} \Big|_{\frac{R_2}{3}}^{R_2} - 0,001230 \cdot \frac{1}{5} \cdot \frac{R_2^6}{r^5} \Big|_{\frac{R_2}{3}}^{R_2} + \right. \\ \left. - 0,00003122 \cdot \frac{1}{7} \cdot \frac{R_2^8}{r^7} \Big|_{\frac{R_2}{3}}^{R_2} - 0,0000006619 \cdot \frac{1}{9} \cdot \frac{R_2^{10}}{r^9} \Big|_{\frac{R_2}{3}}^{R_2} - 0,00000001416 \cdot \frac{1}{11} \cdot \frac{R_2^{12}}{r^{11}} \Big|_{\frac{R_2}{3}}^{R_2} \right]$$

or

$$\int_{\frac{R_2}{3}}^{R_2} (\Theta\Theta)_{\Theta=0} dr = 1,00009 \frac{P}{\pi}$$

To explain the fact that the summation of normal stresses in the plane of the vertical radial cross section is not equal to zero, it is necessary to consider the distribution of stresses at the point of application of the force P (Fig. 16). By diminishing the length of the radius r_1 , with notations as shown in Fig. 13, the value of stresses acting in the direction of this radius will be increased, as follows from the formula (53b). Finally at the point of application of the force P , the value of radial stress becomes infinitely large. Thus from the formulas (53a-b-c) it is sufficient to take only the one term:

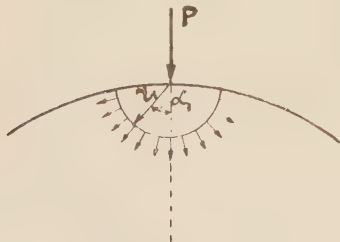


Fig. 16.

$$\sigma_2 = - \frac{2P \cos \alpha_1}{\pi r_1} \Big|_{r_1 \rightarrow 0}$$

since the other two terms σ_1 and σ_3 may be omitted as giving comparatively small values of stresses.

The summation of horizontal components of σ_2 stresses gives the value:

$$\int_0^{\frac{\pi}{2}} - \frac{2 P \cos \alpha_1 \cdot \sin \alpha_1}{\pi r_1} r_1 \cdot d\alpha_1 = - \frac{P}{\pi}$$

and this component together with the force $\int_{\frac{R_2}{3}}^{R_2} (\Theta\Theta)_{\Theta=0} dr$ gives the zero value of the normal force which agrees with the conditions of equilibrium.

Stress diagram for $\Theta\Theta$ stress along vertical and horizontal radial cross sections
(with $\Theta = 0$ and $\Theta = \frac{\pi}{2}$).

For both cases the stress formula may be expressed in the form:

$$(\Theta\Theta) = \frac{2 P}{\pi R_2} \cdot n$$

where values of factor n are given in the following tables.

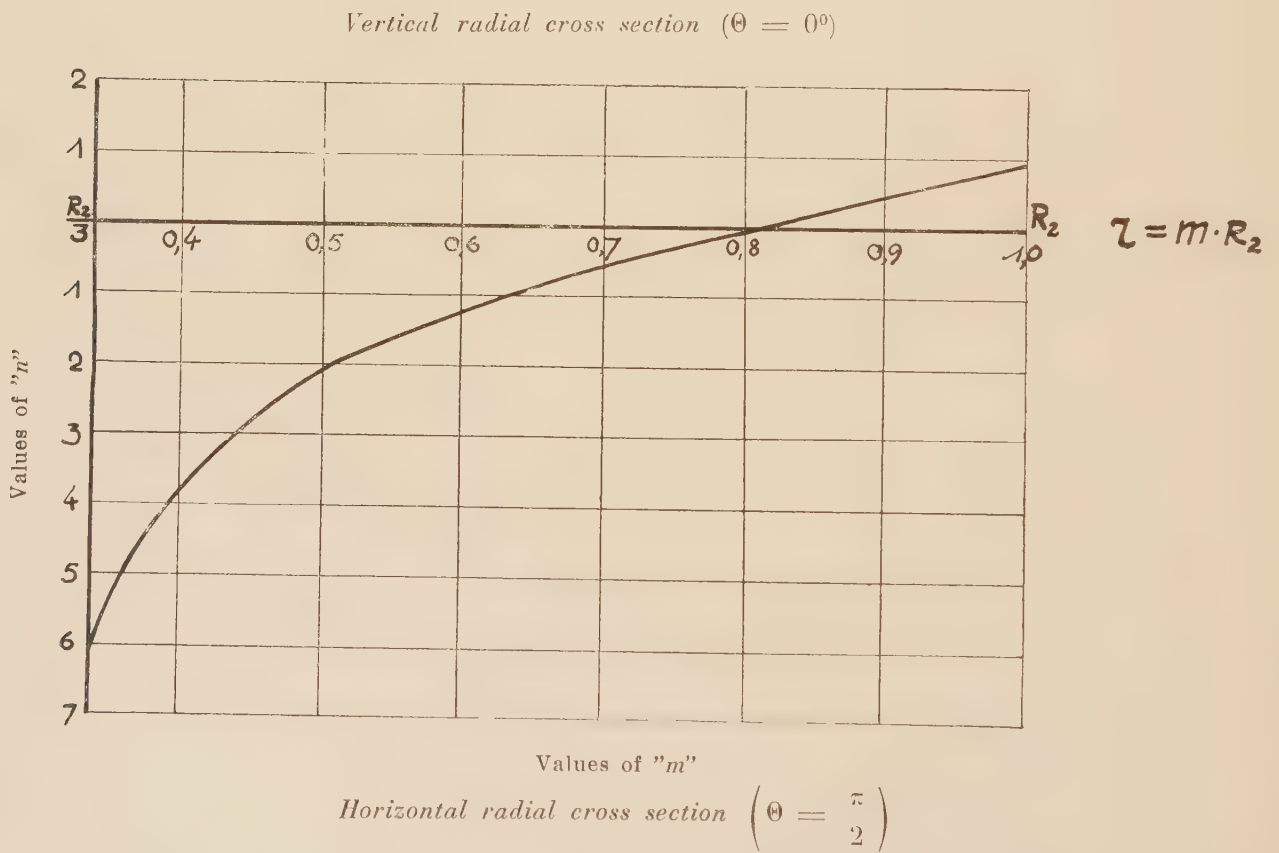
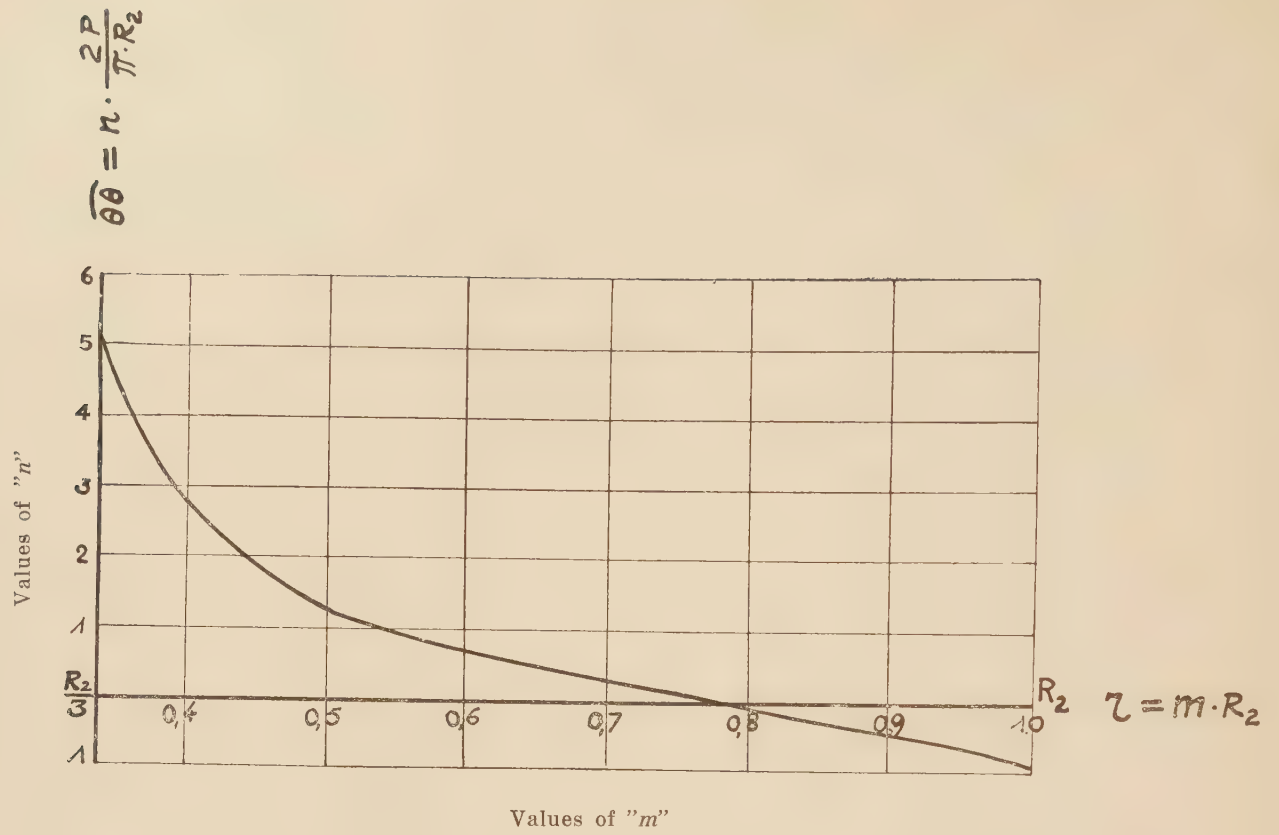
Vertical radial cross section ($\Theta = 0$)
values of n :

$r =$	$\frac{R_2}{3}$	$0,4 R_2$	$0,5 R_2$	$0,6 R_2$	$0,7 R_2$	$0,8 R_2$	$0,9 R_2$	R_2
$n =$	5,4591	2,6722	1,2410	0,6495	0,2659	— 0,0716	— 0,4154	— 0,8863

Horizontal radial cross section ($\Theta = \frac{\pi}{2}$)
values of n :

$r =$	$\frac{R_2}{3}$	$0,4 R_2$	$0,5 R_2$	$0,6 R_2$	$0,7 R_2$	$0,8 R_2$	$0,9 R_2$	R_2
$n =$	— 6,0602	— 3,7649	— 2,1198	— 1,2129	— 0,5623	— 0,0156	0,4850	0,9656

The corresponding curves are shown in Fig. 17.



Distribution of $\sigma_{\theta\theta}$ stress in circular ring under the action of two opposite forces in diametrical direction.

Fig. 17.

V. Appendix.

Stress distribution in circular arches under the action of pure bending.

Assume a circular ring with an internal moment M acting on each cross section (Fig. 18). Symbols a and b represent the length of the inner and outer radii of the arch. In this case the stress function;

$$\Phi = A \log r + Br^2 \log r + Cr^2 + D \quad (64)$$

gives the values of normal stresses acting in radial and tangential directions, while the shearing stress is equal to zero due to the symmetry of loading.

$$\text{Radial stress} \quad \widehat{rr} = \frac{1}{r} \cdot \frac{d\Phi}{dr}$$

$$\text{Tangential stress} \quad \widehat{\theta\theta} = \frac{d^2\Phi}{dr^2}$$

or after substituting the value of Φ as given above:

$$\widehat{rr} = \frac{A}{r^2} + 2B \log r + B + 2C$$

$$\widehat{\theta\theta} = -\frac{A}{r^2} + 2B \log r + 3B + 2C$$

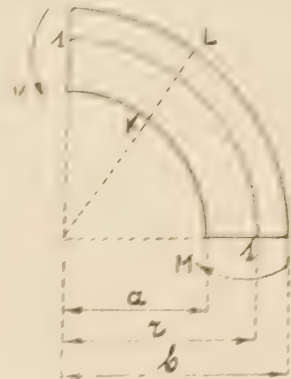


Fig. 18

where values of the parameters A , B , C , may be found from the external loading conditions:

- 1) On the inner surface $\widehat{rr} = 0$ when $r = a$
- 2) On the outer surface $\widehat{rr} = 0$ when $r = b$
- 3) Internal bending moment $= M = \int_a^b \widehat{\theta\theta} r dr$

$$1) \quad \frac{A}{a^2} + 2B \log a + B + 2C = 0$$

$$2) \quad \frac{A}{b^2} + 2B \log b + B + 2C = 0$$

$$3) \quad A \log a - \log b + B (a^2 \log a - b^2 \log b) + C (a^2 - b^2) = M$$

The following notations are used to be substituted into (65):

$$\frac{b}{a} = k \quad \text{where} \quad k > 1 \quad \text{and} \quad \frac{r}{a} = m \quad \text{where} \quad 1 < m < k$$

The values of stresses:

$$\widehat{rr} = \frac{4 \left(\log m + k^2 \log \frac{k}{m} - \frac{k^2}{m^2} \log k \right)}{4k^2 \log k^2 - (1 - k^2)^2} \cdot \frac{M}{a^2}$$

$$\widehat{\theta\theta} = \frac{4 \left(1 - k^2 + \log m + k^2 \log \frac{k}{m} + \frac{k^2}{m^2} \log k \right)}{4k^2 \log k^2 - (1 - k^2)^2} \cdot \frac{M}{a^2}$$

To compare the solutions obtained with elementary solutions several numerical examples may be taken:

$$\frac{b}{a} = 3, \quad \frac{b}{a} = 2, \quad \frac{b}{a} = 1,3.$$

In the following tables the values of factors of $\frac{M}{a^2}$ in the formulas (65a) are given.

$$1) \quad \frac{b}{a} = 3$$

$m =$	1,0	1,2	1,4	1,6	1,8	2,0	2,2	2,4	2,6	2,8	3,0
factors of $\widehat{rr} =$	0,000	-0,304	-0,419	-0,441	-0,415	-0,364	-0,299	-0,227	-0,152	0,076	0,000
factors of $\widehat{\Theta\Theta} =$	-2,292	-1,420	-0,825	-0,387	-0,046	+0,231	+0,463	+0,662	+0,836	+0,990	+1,130

$$2) \quad \frac{b}{a} = 2$$

$m =$	1,0	1,1	1,2	1,3	1,4	1,5	1,6	1,7	1,8	1,9	2,0
factors of $\widehat{rr} =$	0,000	-0,595	-0,915	-0,051	-1,062	-0,987	-0,852	-0,674	-0,468	-0,241	0,000
factors of $\widehat{\Theta\Theta} =$	-7,755	-5,418	-3,507	-1,908	-0,542	+0,645	+1,689	+2,620	+3,458	+4,220	+4,917

$$3) \quad \frac{b}{a} = 1,3$$

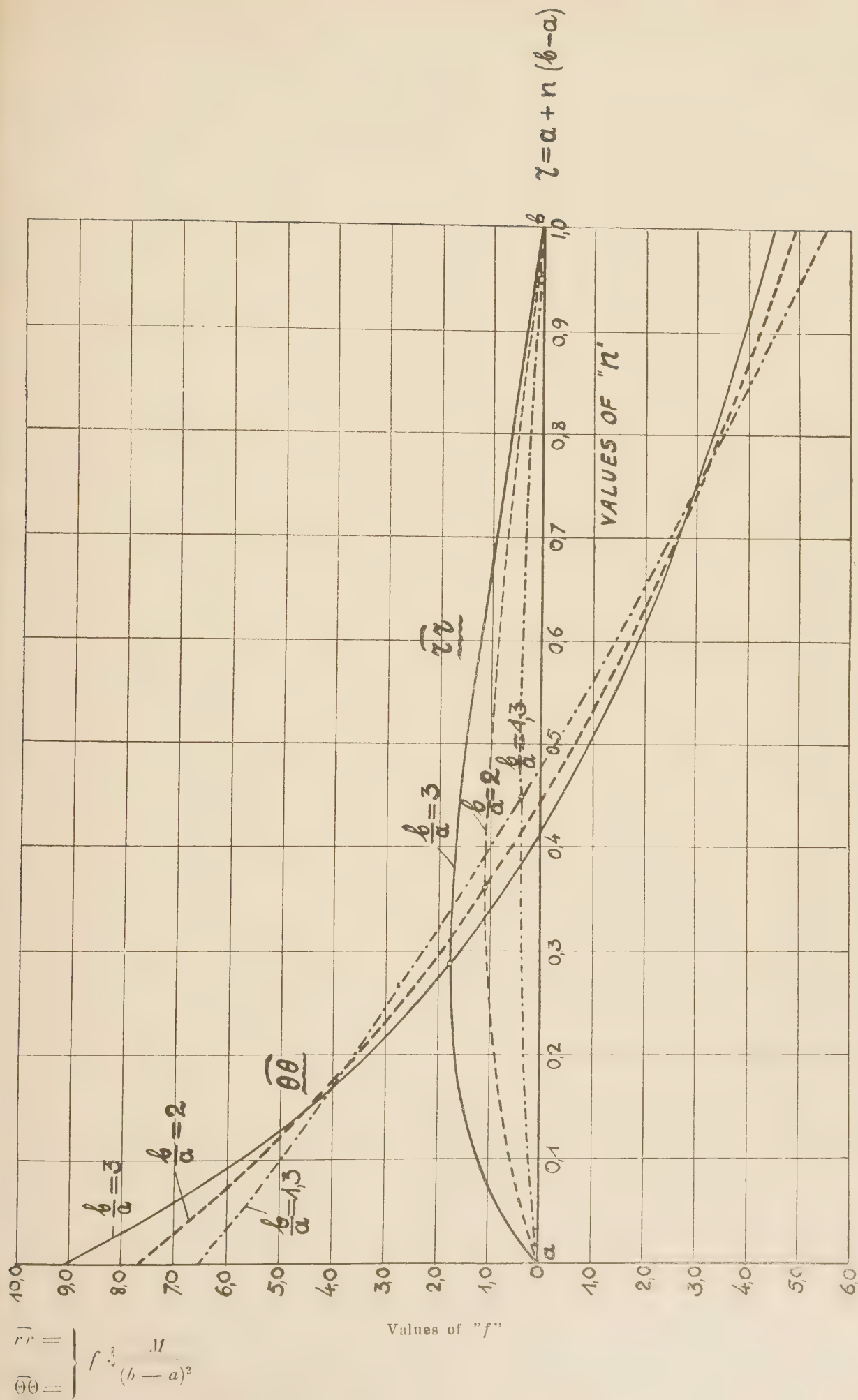
$m =$	1,00	1,04	1,08	1,10	1,14	1,18	1,20	1,24	1,28	1,30
factors of $\widehat{rr} =$	0,000	-2,372	-3,769	-4,154	-4,383	-3,991	-3,593	-2,450	-0,904	0,000
factors of $\widehat{\Theta\Theta} =$	-73,053	-50,589	-29,859	-20,074	-1,546	-15,728	+23,941	+39,596	+54,314	+61,353

To find $(\widehat{rr})_{\max}$ and $(\widehat{\Theta\Theta})_{\max}$ the condition, $\frac{d(\widehat{rr})}{dm} = 0$, must be satisfied:

$$1) \quad k = 3 \quad (\widehat{rr})_{\max} = -0,4414 \frac{M}{a^2} \quad \text{when} \quad m = 1,572$$

$$2) \quad k = 2 \quad (\widehat{rr})_{\max} = -1,070 \frac{M}{a^2} \quad \text{,,} \quad m = 1,360$$

$$3) \quad k = 1,3 \quad (\widehat{rr})_{\max} = -4,393 \frac{M}{a^2} \quad \text{,,} \quad m = 1,134$$



Stress distribution in circular arches under the action of pure bending.

Fig. 19.

It's easy to find from the formula (65a) that max. values of $\widehat{\Theta\Theta}$ stress correspond to extreme fibers in cross section. Comparison of maximum values of $\widehat{rr}$ and $\widehat{\Theta\Theta}$ stresses gives the following results:

- 1) $k = 3$ $(\widehat{rr})_{\max} = 0,193 (\widehat{\Theta\Theta})_{\max}$
- 2) $k = 2$ $(\widehat{rr})_{\max} = 0,138 (\widehat{\Theta\Theta})_{\max}$
- 3) $k = 1,3$ $(\widehat{rr})_{\max} = 0,060 (\widehat{\Theta\Theta})_{\max}$

This indicates the small error of elementary solutions where $\widehat{rr}$ stresses are neglected, especially in arches with small curvature. To compare the exact solutions with approximate formulas usually applicable in elementary calculations, the following are values of stresses for the previous cases, found from the linear, hyperbolic and exact formulas.

k	Linear stress distribution	Hyperbolic stress distribution		Formula (65a)	
1,3	$\pm 66,667$	+ 61,268	- 72,982	+ 61,353	- 73,053
2	$\pm 6,000$	+ 4,863	- 7,725	+ 4,917	- 7,755
3	$\pm 1,500$	+ 1,095	- 2,285	+ 1,130	- 2,292

Here is seen the great accuracy of hyperbolic solutions. In Fig. 19 is shown the distribution of $\widehat{rr}$ and $\widehat{\Theta\Theta}$ stresses for the previous cases. The length of cross section is taken as constant:

$$b - a = \text{const}$$

and values of stresses are assumed in the form:

$$\left. \begin{array}{l} \widehat{rr} = \\ \widehat{\Theta\Theta} = \end{array} \right\} f \cdot \frac{M}{(b - a)^2}$$

where the magnitude of factor "f" is obtained from the previous table by multiplication:

- 1) by 4 ~ when $\frac{b}{a} = 3$, 2) by 1 ~ when $\frac{b}{a} = 2$, 3) by 0,09 ~ when $\frac{b}{a} = 1,3$.

Value of factor n on the horizontal axis is given by the formula:

$$n = \frac{m - 1}{k - 1}$$

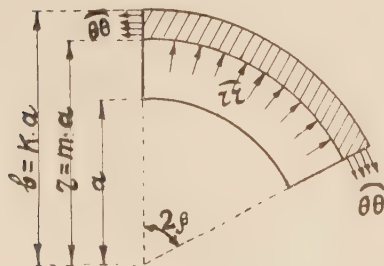


Fig. 20.

To check the answers of the problem, the condition of equilibrium of an element (Fig. 20) will be considered:

$$2 \int_r^b \widehat{\Theta\Theta} \cdot \sin \beta \cdot dr = - 2 r \cdot \sin \beta \cdot \widehat{rr} \quad . \quad . \quad (66)$$

or after substitution of values from formula (65a)

$$\int_m^k \left[\log m \cdot (1 - k^2) + \frac{k^2 \log k}{m^2} + k^2 \log k - k^2 + 1 \right] dm \equiv -m \left[\log m \cdot (1 - k^2) + \right. \\ \left. - \frac{k^2}{m^2} \log k + k^2 \log k \right]$$

This agrees very closely.

The extreme points of $\widehat{rr}$ stresses are marked in Fig. 19 as P_1, P_2, P_3 . It is easy to check that corresponding curves of $\widehat{rr}$ and $\widehat{\theta\theta}$ stresses intersect each other in these points; for

$$\frac{d(\widehat{rr})}{dm} = 0 \quad \text{when} \quad m^2 = \frac{2k^2 \log k}{k^2 - 1}$$

and this is the condition when $\widehat{rr} = \widehat{\theta\theta}$

Position of the neutral axis of $\widehat{\theta\theta}$ stresses.

As follows from the condition of equilibrium (formula 66) it is easy to see that the radial compressive force in any arc of a radial cross section and for any given angle β is proportional to the value of $m \cdot \widehat{rr}$. Thus the maximum value of this force will be found from the condition that the first derivative of $m \cdot \widehat{rr}$ with respect to m is equal to zero,

$$\frac{d}{dm} (m \cdot \widehat{rr}) = 0$$

$$\text{or} \quad \frac{d}{dm} \left[m \cdot \log m \cdot (1 - k^2) + m k^2 \log k - \frac{k^2 \log k}{m} \right] = 0$$

$$\text{which gives} \quad 1 - k^2 + \log m + k^2 \log \frac{k}{m} + \frac{k^2}{m^2} \log k = 0 \quad (67)$$

But the last polynomial is exactly the numerator in the expression for $\widehat{\theta\theta}$ stress, which means that the points of maximum value of the radial force are on the neutral axis of tangential stress.

Shearing stress distribution in circular arches under the action of a force P in a radial cross section.

The stress function for this case may be written in the form:

$$\Phi = \left(A r^3 + \frac{B}{r} + C r + D r \cdot \log r \right) \sin \theta \quad (68)$$

The values of internal stresses may be found from formula (9) with $\gamma_w = 0$

$$\text{thus} \quad \widehat{rr} = \frac{1}{r} \cdot \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 \Phi}{\partial \theta^2}$$

$$\widehat{\theta\theta} = \frac{\partial^2 \Phi}{\partial r^2}$$

$$\widehat{r\theta} = - \frac{\partial}{\partial r} \left(\frac{1}{r} \cdot \frac{\partial \Phi}{\partial \theta} \right)$$

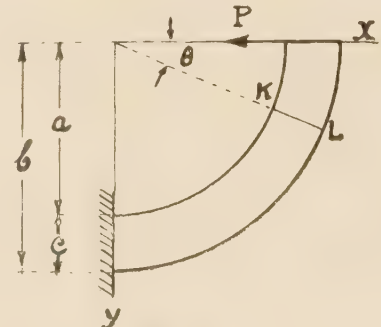


Fig. 21.

or after the substitution of the value for Φ this gives:

$$\widehat{r\Theta} = - \left(2 A r - \frac{2 B}{r^3} + \frac{D}{r} \right) \cos \Theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

The values of the parameters A, B, D may be found from boundary conditions, which give three additional relations:

$$1) \quad \text{for } \Theta = 0, \quad \int_a^b (\widehat{r\Theta})_{\Theta=0} dr = P$$

$$2) \quad \text{for } r = a, \quad \widehat{rr} = 0,$$

$$3) \quad \text{for } r = b, \quad \widehat{rr} = 0$$

or after the substitution of the value for $\widehat{r\Theta}$ from formula (69) these relations will read:

$$\left. \begin{aligned} 1) \quad & A (a^2 - b^2) + B \left(\frac{1}{a^2} - \frac{1}{b^2} \right) + D \cdot \log \frac{a}{b} = P \\ 2) \quad & 2 A a - \frac{2 B}{a^3} + \frac{D}{a} = 0, \quad 3) \quad 2 A b - \frac{2 B}{b^3} + \frac{D}{b} = 0 \end{aligned} \right\} \quad . \quad . \quad (70)$$

The following notations are used to substitute into (69)

$$b = k \cdot a \quad \text{and} \quad r = m \cdot a$$

$$\text{then} \quad \widehat{r\Theta} = \frac{m^2 (m^2 - k^2 - 1) + k^2}{m^3 [(k^2 - 1) - (k^2 + 1) \log k]} \cdot \frac{P \cdot \cos \Theta}{a} \quad . \quad . \quad . \quad . \quad . \quad (71)$$

Maximum value of shearing stress.

The condition $\frac{\partial (\widehat{r\Theta})}{\partial m} = 0$ applied to formula (71) gives:

$$m^4 + m^2 (k^2 + 1) - 3 k^2 = 0 \quad \text{and} \quad m = \sqrt{\frac{-(k^2 + 1) + \sqrt{k^4 + 14 k^2 + 1}}{2}} \quad (72)$$

The inflection point corresponds to $\frac{\partial^2 (\widehat{r\Theta})}{\partial m^2} = 0$, which gives:

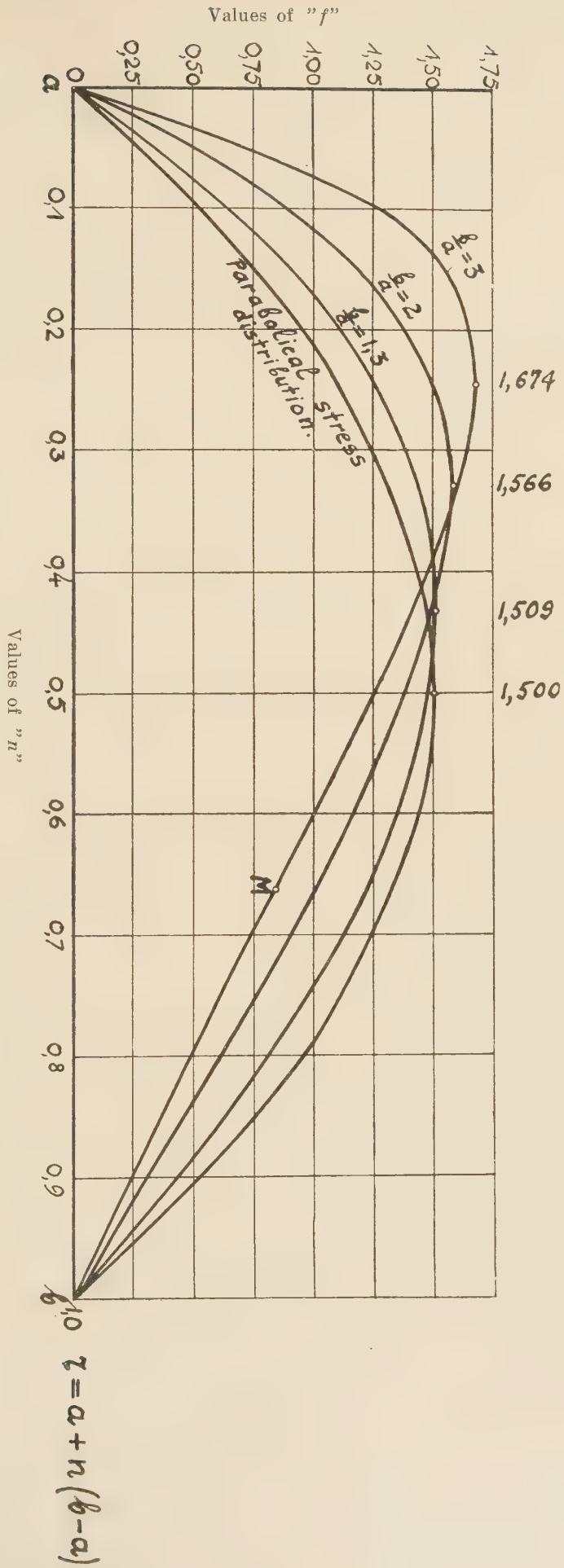
$$m = \sqrt{\frac{6 k^2}{k^2 + 1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

The limit value for m applied to formula (73) shows that when shearing stress diagram contains the inflection point

$$m \leq k, \quad \sqrt{\frac{6 k^2}{k^2 + 1}} \leq k; \quad \text{thus} \quad k \geq \sqrt{5}.$$

For rings with $b < \sqrt{5} \cdot a$ no inflection point exists in the diagram, as is shown in Fig. 22.

$$r(\theta) = f \cdot \frac{P \cdot \cos \theta}{b - a}$$



Shearing stress distribution in the circular bar under the action of radial force applied at the end.

Fig. 22.

Some special cases with 1) $k = 3$, 2) $k = 2$, 3) $k = 1,3$.

1) $k = 3$. The inflection point lies at $m = \sqrt{5,4} = 2,324$ (in Fig. 22 marked as M). The maximum shear found from the exact formula (72) corresponds to

$$m = 1,487, \quad \text{and} \quad (\widehat{r\Theta})_{\max} = 0,837 \frac{P \cos \Theta}{a}$$

From the approximate formula, the max. shear corresponds to

$$m = 2,0, \quad \text{and} \quad (\widehat{r\Theta})_{\max} = 0,750 \frac{P \cos \Theta}{a}$$

2) $k = 2$. No inflection point exists. From the exact formula, the max. shear corresponds to

$$m = 1,331, \quad \text{and} \quad (\widehat{r\Theta})_{\max} = 1,566 \frac{P \cos \Theta}{a}$$

From the approximate formula, the max. shear corresponds to

$$m = 1,500, \quad \text{and} \quad (\widehat{r\Theta})_{\max} = 1,500 \frac{P \cos \Theta}{a}$$

3) $k = 1,3$. No inflection point exists. From the exact formula, the max. shear corresponds to

$$m = 1,1304, \quad \text{and} \quad (\widehat{r\Theta})_{\max} = 5,030 \frac{P \cos \Theta}{a}$$

From the approximate formula, the max. shear corresponds to

$$m = 1,150, \quad \text{and} \quad (\widehat{r\Theta})_{\max} = 5,000 \frac{P \cos \Theta}{a}$$

In general formula (71) may be written in the form:

$$\widehat{r\Theta} = d \cdot \frac{P \cos \Theta}{a} \quad \dots \dots \dots (71a)$$

Values of coefficients d are given in the following tables.

1) For $k = 3$

$m =$	1,0	1,2	1,4	1,6	1,8	2,0	2,2	2,4	2,6	2,8	3,0
$d =$	0	0,645	0,825	0,821	0,741	0,628	0,502	0,374	0,246	0,121	0

2) For $k = 2$

$m =$	1,0	1,1	1,2	1,3	1,4	1,5	1,6	1,7	1,8	1,9	2,0
$d =$	0	0,905	1,400	1,558	1,532	1,392	1,178	0,917	0,627	0,319	0

3) For $k = 1,3$

$m =$	1,00	1,04	1,08	1,10	1,14	1,18	1,20	1,24	1,28	1,30
$d =$	0	2,800	4,389	4,805	5,009	4,510	4,039	2,727	0,997	0

In Fig. 22 are shown diagrams of shearing stresses in the cross section $K - L$ (Fig. 21) for the previous three cases and for parabolical stress distribution. The length of radial cross section is taken as constant $b - a = \text{const.} = c$. Then in formula (71a):

$$a = \frac{c}{2} \quad \text{for} \quad k = 3$$

$$a = c \quad \text{,,} \quad k = 2$$

$$a = \frac{10}{3} c \quad \text{,,} \quad k = 1,3$$

The length of any inner radius is, $r = a + n(b - a)$ where $n = \frac{m-1}{k-1}$. The parabolical stress distribution agrees very closely with the exact stress distribution if $\frac{b}{a}$ approaches unity.

Conclusion.

An interesting and important result of the previous considerations is the possibility to use the formulas for circular elastic rings under the action of body forces and a single concentrated force, as given in § III — for rings with any system of normal, external forces acting in equilibrium (without body forces).

As follows from the problem of the ring with two concentrated forces, the formulas (45, 46, 47) are general solutions for the problem with any number of these forces. Imaginary body forces corresponding to each force successively, may be used. After the summation of all items obtained, the final formula expresses the solution of the problem considered.

